# Introduction

▪ **Optimization**

- Finding the best solution among set of all feasible solutions.
- Deals with the problem of minimizing or maximizing a function with several variables subject to some constraints.
- Everyone, almost daily, solves optimization problems in informal ways by using mental models.
- Optimization plays a central rule in operation research, management science, and engineering.

# Optimization Techniques

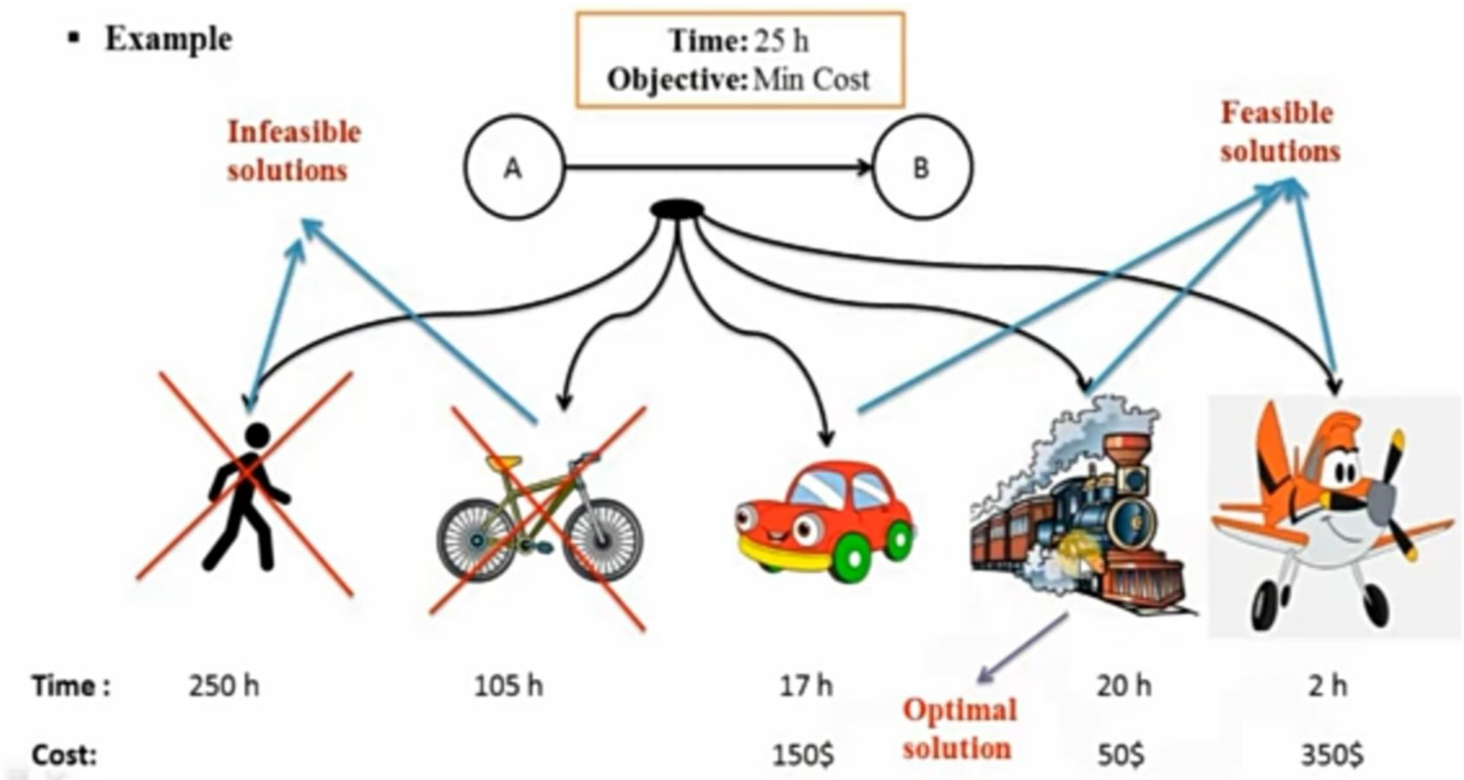
• **Some objectives in optimization problems may be**

- Minimize cost
- Maximize profit
- Planning production
- Increase process efficiency

■ **Some typical applications from different engineering disciplines:**

- Deciding on the most effective allocation of limited resources.
- Choosing control variables that will cause a system to behave as desired.

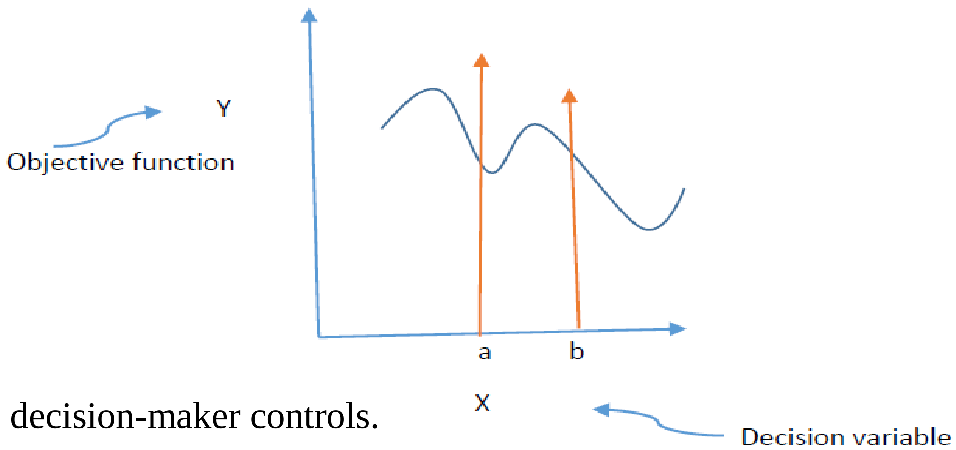
• **Example**




## Optimization Terminologies

### Objective Function

✓ The function that it is desired to maximize or minimize.




### Decision Variable

✓A quantity that the decision-maker controls.

✓ All decision variables should be independent of each other.

- **Local minimum (maximum)**

- ✓ A point where the function value is smaller (greater) than or equal to the value at nearby points.
- **Global minimum (maximum)**

✓ A point where the function value is smaller (greater) than or equal to the value at all other feasible points.

## Optimization Terminologies

**Search Space** a function $f(\boldsymbol{X})$ defined at a closed interval [a, b].

**Unimodal function**
✓ A function has only one peak (maximum, concave) or valley (minimum, convex) in a given interval.

## Constraint

✓ The constraints represent some functional relationships among the design variables and other design parameters satisfying certain **physical phenomenon** and certain resource limitations**.**

✓ Figure below shows the feasible region in a two- dimensional design space.

✓ The set of values of X that satisfy the equation gi (X) = **0** forms a hyper surface in the design space and is called a constraint surface.

• **Classification of the Optimization Problems**

In an optimization problem, the types of mathematical relationships between the objective and constraints and the decision variables determine how hard it is to solve, the solution methods or algorithms that can be used for optimization, and the confidence you can have that the solution is truly optimal.

■ **Classification of the Optimization Problems**

. Classification based on the number of decision variables**:**

✓Single variable

✓Multi variable

- Classification based on the existence of constraints**:**

✔ Unconstrained optimization

✓ Constrained optimization

- Classification based nature of the equations involved:

✓ Linear optimization

✓ Nonlinear optimization

**Classification of the Optimization Problems**

- Classification based on number of objective function:

✓ Single objective

✓ Multi objective

- Classification based on the deterministic nature of the variables

✓ Deterministic ✓ Stochastic

- Classification based on time-dependent of the problem

✓ Static ✓ Dynamic

**Optimization Techniques**

Classical Optimization Techniques

✓ Single Variable Optimization

✓ Multivariable Optimization

> Multivariable Optimization with no constrained

> Multivariable Optimization with Equality Constraints

> Multivariable Optimization with Inequality Constraints Optimization Techniques Introduction Numerical Optimization Techniques

✓ Elimination Methods ✓ Fibonacci method

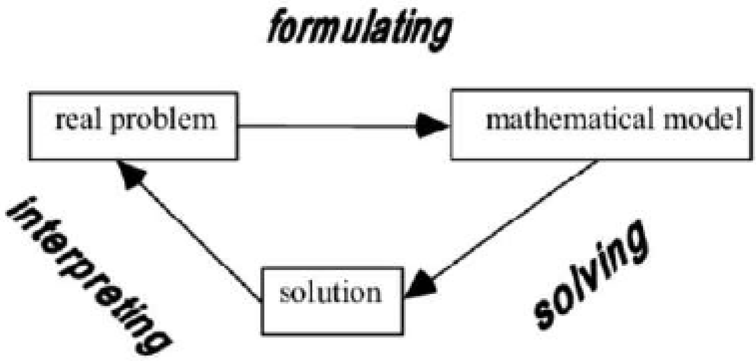
✓ Golden section method ✓ Interpolation Methods

✓ Newton Method

## • Mathematical Modelling of Optimization Problem

• A collection of mathematical equations that help to explain a system and to study the effects of different components, and to make predictions about behavior . Several approximations and assumption often made It is often possible to construct more than one form of an mathematical model that represents the same problem equally accurately.

Investigate the class of problem encountered.



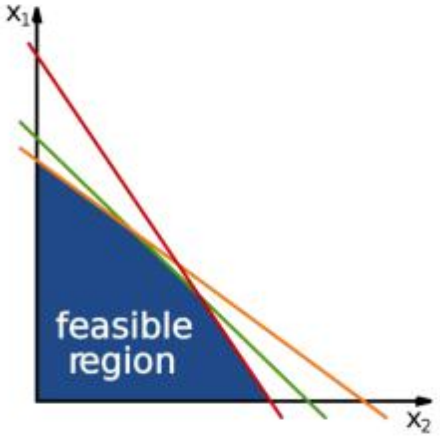

▪ **Linear** Programing

If the objective functions and all the constraint functions in the optimization problem is linear, the problem is called a ***linear*** **programming *problem***

Min (Max) f(x2,x2,...,xg)

Subject to the constraints

g(X1, X2,….. Xg)
h(X1, X2,….. Xg)



• **Linear Programing. Example**. A Company has requested a manufacturer to produce formal and sports jackets. For materials, the manufacturer has 750 $m^2$ of cotton textile and 1000 $m^2$ of polyester. Every formal jackets needs 1 $m^2$ of cotton and 2 $m^2$ of polyester. Every sport jacket needs 1.5 m2 of cotton and 1 m2 of polyester. The price of the formal jackets is $50 and the sport jacket is $40. The Company want to know what the number of formal jackets and sport jackets must request so that these items obtain a maximum sale. Formulate the problem as an optimization problem.

## ■ Solution.

Decision Variables
X1 = number of formal jackets x2 = number of sport jackets

Objective Function Max $f\,(x1,\, \mathrm{x2}) = 50x1 + 40x2$

Subject to Constraint
X1+1.5x2750 → Cotton textile

2x1 + x21000 → Polyester

• Goal Programming. Example. Company produces two products (A and B). The two products

require a one hour labor work. The available labor work **is 300** hours per month. The demand of product

A and product B in a month are 140 and 200 respectively. The profit from the sale of product A is 600$

and product B is 200$. The manager has set the following goals.

✓ Pl: Avoid any underutilization of normal production capacity.

✓ P2: Sell maximum possible units of product A and B.

✓P3: Minimize the overtime operation of the plant as much as possible.

Formulate the given problem as an optimization problem?

• Solution
Decision Variables
X1 = number of product A x2 = number of product B

Goal **1** P2: $x1 + x2 + d1^{+} + d2^{-} = 300$

Unwanted deviations Goal 1→$d1^{-}$

Goal **2** → P2: $x1 + d2^{-} = 140$

Goal **2** → P2: $x2 + d3^{-} = 200$

Unwanted deviations Goal 2→$3d2^{-} + d3^{-}$

Goal **3** → P3: $x1 + x2 + d1^{+} + d1^{-} = 300$

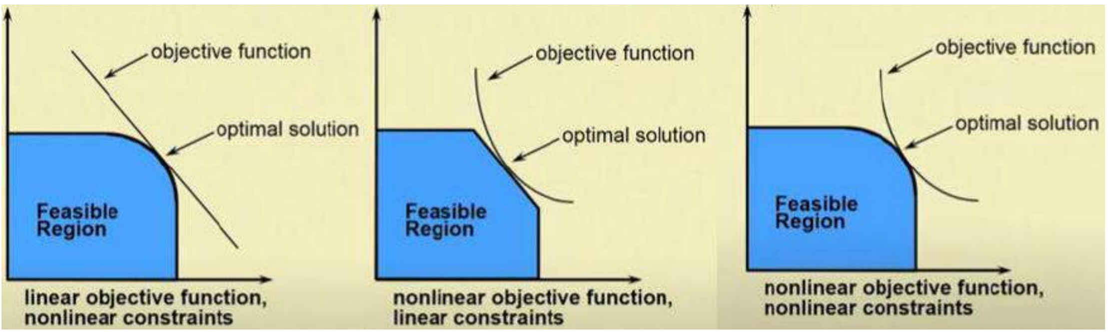
Minimize p2(d1-),p2(3d2- +d3-),p3(d1+)

Subject to: $x1 + x2 + d1^{+} + d2^{-} = 300$ : $x1 + d2^{-} = 140$ : $x2 + d3^{-} = 200$

All variables $\geq 0$

## ■ Nonlinear Programming

If any of the functions among the objective and constraint functions in the optimization problem is nonlinear; the problem is called a ***nonlinear programming problem***.




▪ **Nonlinear Programing. Example.** A company is planning to introduce two new products: A and B. It is estimated that the price of product A is \$339, and the price of product B is \$399. It also estimated that the cost of product A is \$195 per unit, and the cost of product B is \$225 per unit, plus additional cost of \$400000. In the competitive market the number of sales will affect the sales price. It is estimated that for each product, the sales price drops by one cent for each additional unit sold. Furthermore, sales of the product A will affect sales of the product B and vice versa. It is estimated that the price for the product A will be reduced by an additional 0.3 cents for each product B sold, and the price for product B will decrease for by 0.4 cents for each of product A sold. Formula maximize the profit of the company?

■ Solution.
Decision Variables
X1 = number of product A x2 = number of product B

**Price of** product A →p1= $339-0.01x_1 -0.003x_2$

**Price of** product B → P2 = $399-0.01x_2 -0.004x_1$

Revenue →R = $p1x_1 + p2x_2$

R = $(339-0.01x1 -0.003x_2 )x_1 + (399-0.01x_2 -0.004x_1 )x_2$

R = $339x_1 + 399x_2 – 0.1x_1^2 – 0.1x_2^2 – 0.007x_1x_2$

Profit = R – C

C = 195x1 + 225x2 + 400000

Profit = $339x_1 + 399x_2 – 0.1x_1^2 – 0.1x_2^2 – 0.007x_1x_2$ - 195x1 - 225x2 – 400000

Max. Profit $(x_1, x_2) = 144x_1 + 174 x_2 – 0.01x_1^2 – 0.007x_1x_2 – 225x_2 – 400000$

## *Linear Programming*

## *Special Cases in Graphical Method*

Multiple Optimal Solution

## Example 1

Solve by using graphical method
Max Z = 4x1 + 3x2
Subject to
4x1+ 3x2 ≤ 24
x1 ≤ 4.5
x2 ≤ 6

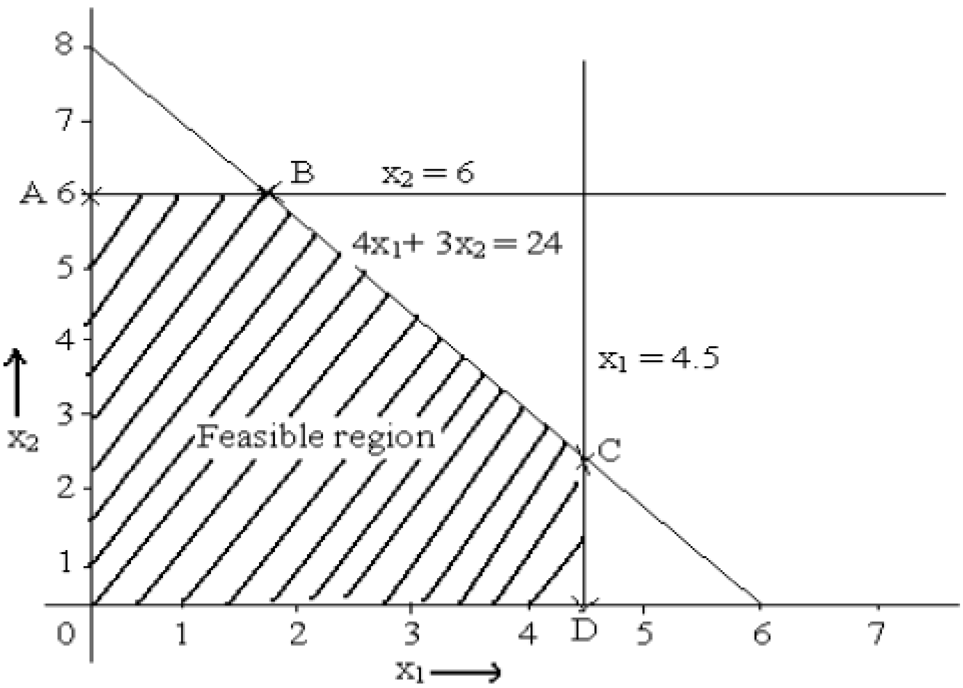
x1 ≥ 0, x2 ≥ 0
**Solution**

- The first constraint 4x1+ 3x2 ≤ 24, written in a form of equation

4x1+ 3x2 = 24, Put x1 =0, then x2 = 8, Put x2 =0, then x1 = 6
The coordinates are (0, 8) and (6, 0)

- The second constraint x1 ≤ 4.5, written in a form of equation x1 = 4.5

The third constraint x2 ≤ 6, written in a form of equation x2 = 6




**Example**: Find the feasible region and the optimal solution of the following

Max $Z = 5x_1 + 4x_2$

Sub. To $2x_1 + 4x_2 \leq 8$

$-2x_1 + x_2 \leq 2$

$3x_2 \leq 9$

$4x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

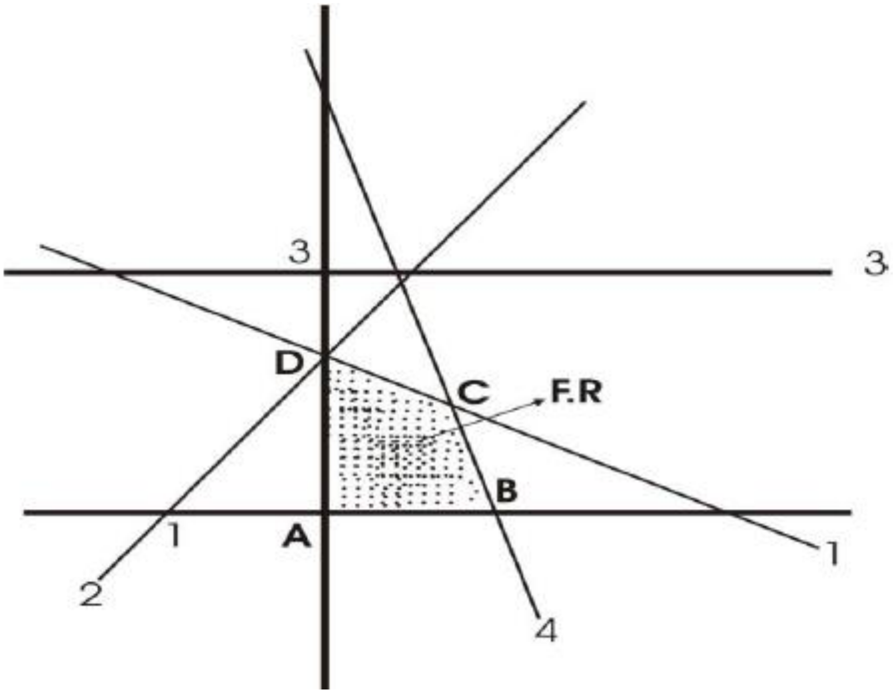
**Solution:**

Let $2x_1+4x_2=8$ … line (1) $\Rightarrow$(0,2), (4,0)

$-2x_1+ x_2 = 2$ … line (2) $\Rightarrow$(0,2), (-1,0)

$3x_2=9$ … line (3)$\Rightarrow$(0,3)

$4x_1+x_2 = 4$ …line (4) $\Rightarrow$(0,4), (1,0)




The optimal solution is:

Z= 9.714 , $x_1 = 4/7$ , $x_2 = 12/7$

**Extreme**

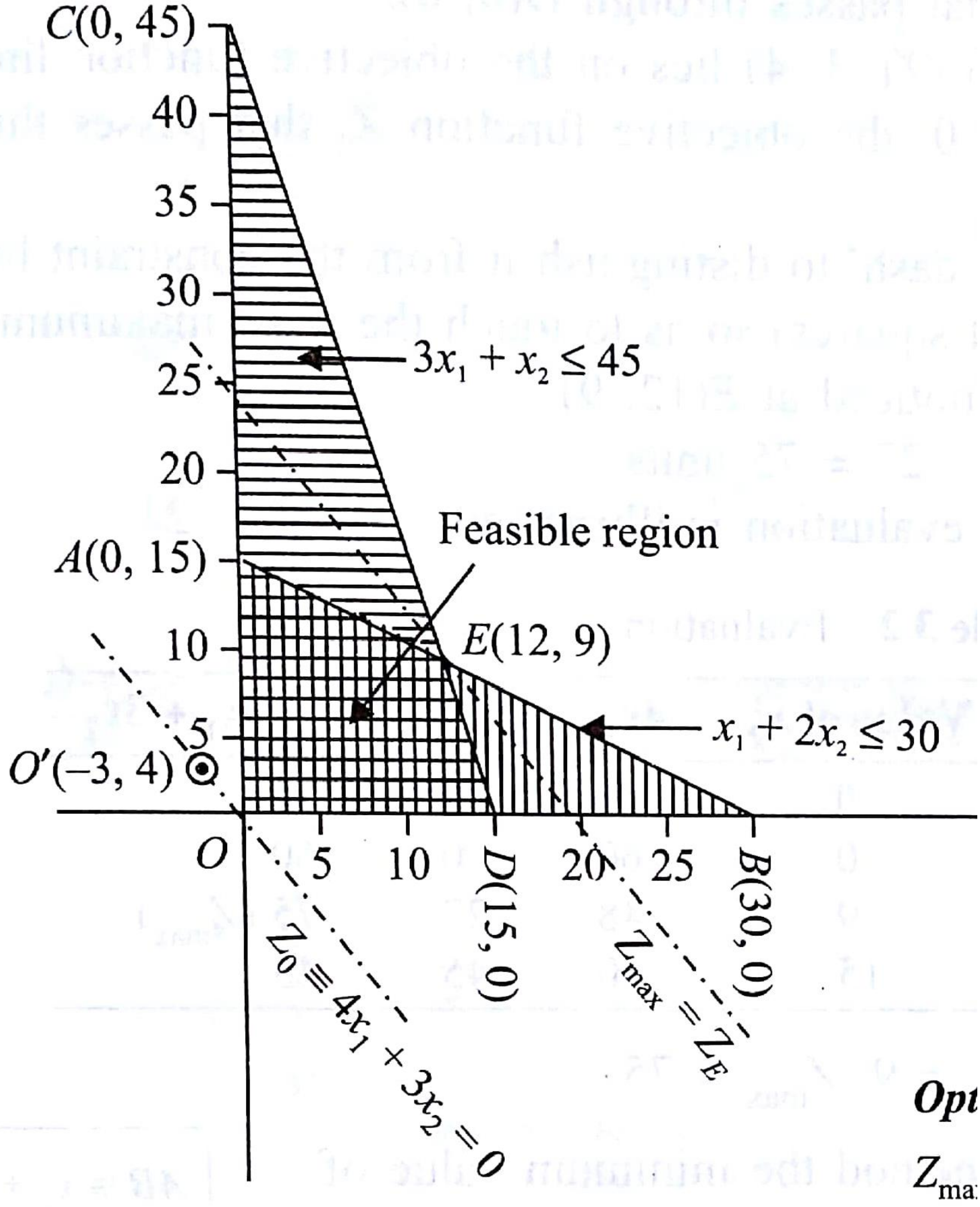
| **Extreme Point** | X1 | X2 | **$Z=5x_1+4x_2$** |
|---|---|---|---|
| A | 0 | 0 | 0 |
| B | 1 | 0 | 5 |
| C | 4/7 | 12/7 | 9.714 |
| D | 0 | 2 | 8 |

Example: Use graphical method to maximize Z=4x1 + 3x2

$x_1+2x_2\leq30$

$3x_1 + x_2 \leq 45$

$x1 \geq 0, x_2 \geq 0$




**Example4.**

Alpha ltd. produces two products X and Y each requiring same production capacity. The total installed production capacity is 9 tones per days. Alpha Ltd. Is a supplier of Beta Ltd. Which must supply at least 2 tons of X & 3 tons of Y to Beta Ltd. Every day. The production time for X and Y is 20 machine hour pr units & 50 machine hour per unit respectively the daily maximum possible machine hours is 360 profit margin for X & Y is Rs. 80 per ton and Rs. 120 per ton respectively. Formulate as a LP model and use the graphical method of generating the optimal solution for determining the maximum number of units of X & Y, which can be produced by Alpha Limited.

**Solution**:

$x_1$ = Number of units (in tons) of Product X.

$x_{2\,=}$ Number of units (in tons) of Product Y.

**Objective function**

Maximize (total profit) $Z = 80\ X_1 + 120\ X_2$

Subject to the constraints:

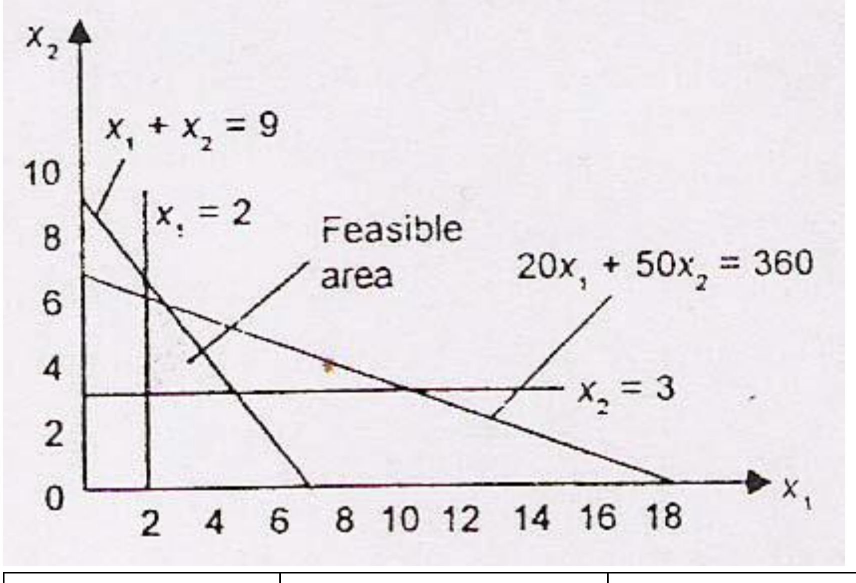
$X_1 + X_2 \leq 9$, $X_1 \geq 2$, $X_2 \geq 3$ (Supply constraint)

$20\ X_1 + 50\ X_2 \leq 360$ (Machine hours' constraint)

and

$X_1 \geq$, $X_2 \geq 0$

X1 + x2 = 9,  x1 = 2,  x2 = 3, 20x1 + 50x2 = 360, and by the coordinates axes.




| Corner Points | Co-ordinates of Corner Points ( $x_1$, $x_2$) | Objective Function $Z=80x_1+20x_2$ | Value |
|---|---|---|---|
| A | (2.6,4) | (80(2) + 120(6.4) | 928 |
| B | (3,6) | (80(3) + 160(6) | 960 |
| C | (6,3) | (80(6) + 160(3) | 840 |
| D | (2,3) | (80(2) + 160(3) | 520 |

The maximum profit (value of Z) of Rs. 960 is found at corner point B i.e., x1=3 and x2 = 6. Hence the company should produce 3 tons of product X and 6 tons of product Y in order to achieve a maximum profit of Rs. 960.

**Example** .

Unique Car Ltd. Manufacturers & sells three different types of Cars A, B, & C. These Cars are manufactured at two different plants of the company having different manufacturing capacities. The following details pertaining to the manufacturing process are provided:

| Manufacturing Plants | Maximum Production (of Cars) | | | Operating cost of Plants |
|---|---|---|---|---|
| | A | B | C | |
| 1 | 50 | 100 | 100 | 2500 |
| 2 | 60 | 60 | 200 | 3500 |
| Demand ( Cards) | 2500 | 3000 | 7000 | |

Using the graphical method technique of linear programming, find the least number of days of operations per month so as to minimize the total cost of operations at the two plants.

**Solution:**

Let $x_1$ = no-of days' plant 1 operates

$x_2$ = no-of days' plant 2 operates.

Objective:

The objective of unique Car Ltd. Is to minimize the operating costs of both its plants.

Minimize $Z = 2{,}500x_1 + 3{,}500x_2$

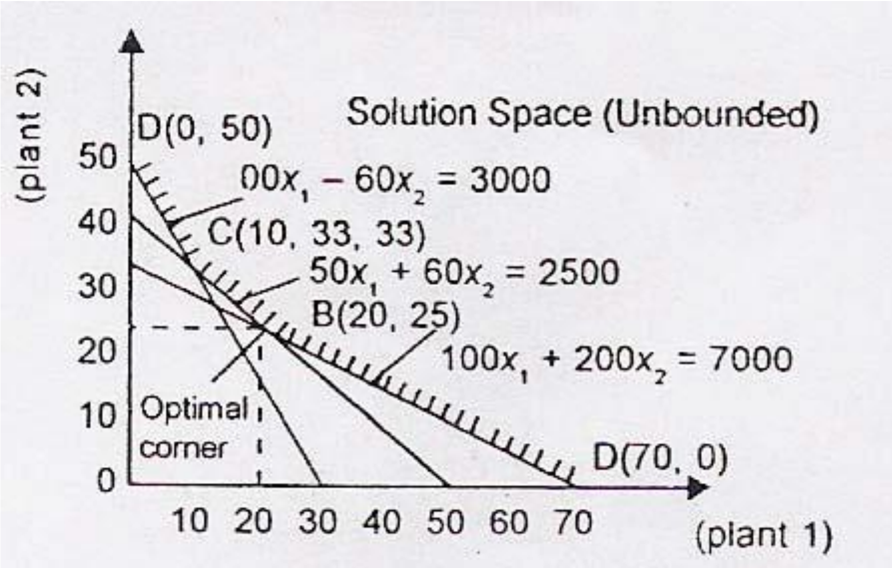
Subject to: $50x_1 + 60x_2 \geq 2500$

$100x_1 + 60x_2 \geq 3000$

$100x_1 + 200x_2 \geq 7000$

$x1, x_2 \geq 0$

Making the graphs of the above constraints:




The solution space lines at the points A,B,C & D Calculating the Optimal solution

| *Points* | *Co-ordinates* | *Objective – func* | *Value* |
|---|---|---|---|
| 0 | (0,0) | Z=0+0 | 0 |
| A | (70,0) | Z=2500x70+0 | 1,75,000 |
| B | (20,25) | Z=2500x20+3500x25 | 1,37,500 |
| C | (10,33.33) | Z=2500x10+3500x33.33 | 1,41,655 |
| D | (0,50) | Z=0+3500x50 | 1,75,000 |

Thus, the least monthly operately cost is at the point B. Where x1=20 days, x2=25 days & operating cost = Rs. 1,37,500.

**Example.**

The chemical composition of common ( table ) salt is sodium chloride ( NACL). Free Flow Salts Pvt. Ltd. Must produce 200 kg of salt per day. The two ingredients have the following cost – profile:

Sodium (Na) - Rs. 3 per kg

Chloride (CL) - Rs. 5 per kg

Using Linear programming find the minimum cost of salt assuming that not more than 80 kg of sodium and at least 60 kg of chloride must be used in the production process.

**Solution**

Formulating as a LP model:

objective Minimize Z=3x+5y

Subject to:

x + y = 200 (Total 200 kg to be produced per day)

x≤80 (Sodium not to exceed 80 kg)

y ≥ 60 (Chloride to be used at least 60 kg)

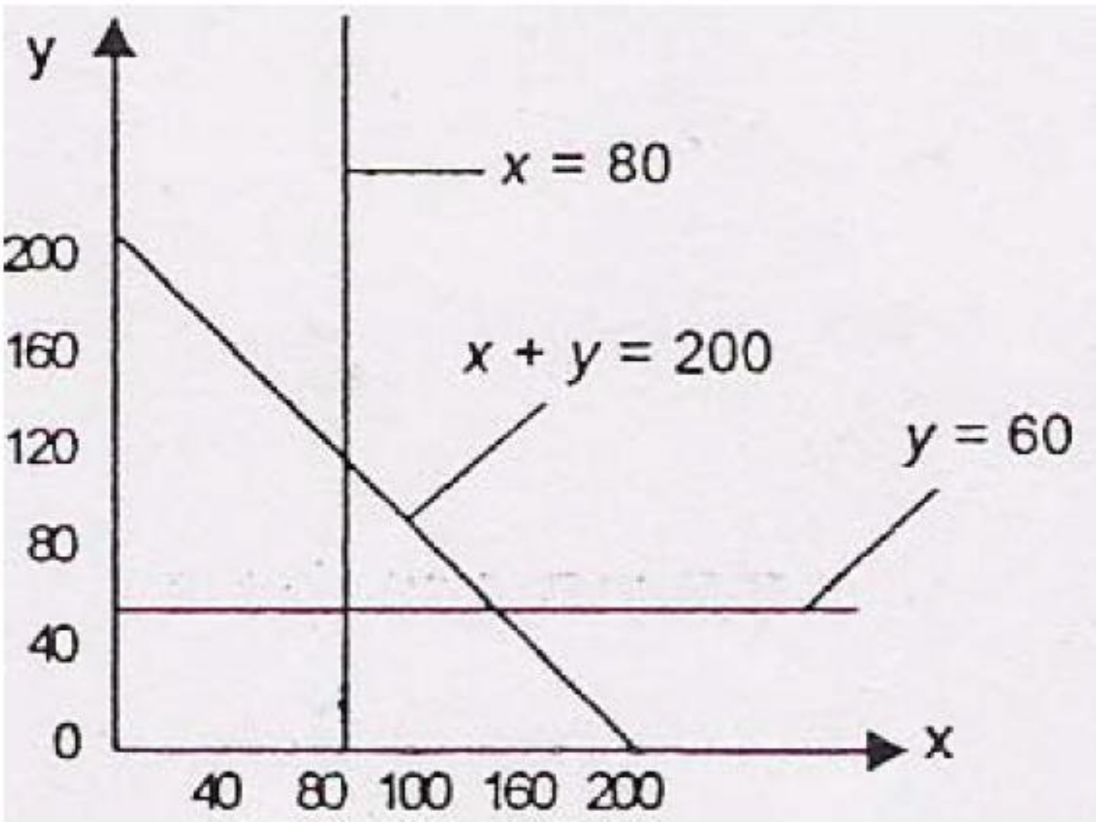
x , y≥ 0

Where

x = Qty of sodium required in the production &

y = Qty of chloride required in the production.

It is clear from the graph that there is no feasible solution area. It has only one feasible point having the co-ordinates (80,120)



# Classical Optimization Techniques

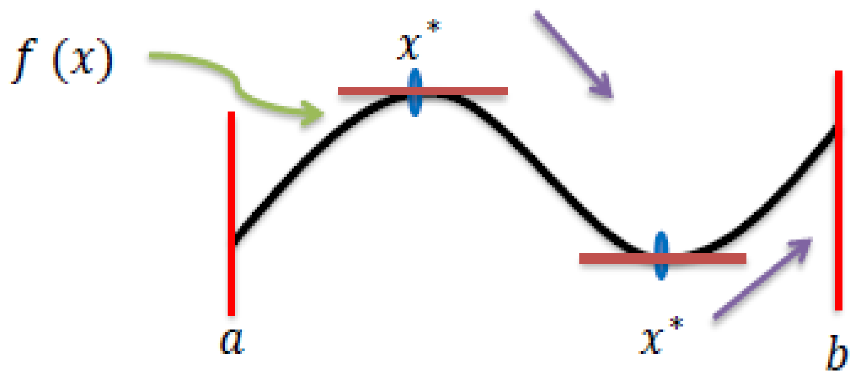
- These methods assume that the function is **differentiable twice** with respect to the design variables and the derivatives are **continuous**.
- These methods are analytical and make use of the techniques of differential calculus in locating the optimum points.
- Since some of the practical problems involve objective functions that are not continuous and/or differentiable, the classical optimization techniques have limited scope in practical applications.

## Single Variable

• The following two theorems:

- **Necessary Condition** If a function $f(x)$ is defined in the interval $a \le x \le b$ and has a relative minimum/maximum at $x = x*$, where $a \le x* \le b$ , and if the derivative $\frac{df(x)}{dx} = \overline{f}$ exists as a finite number at $x = x*$, then $\overline{f}\ (x) = 0$.
- **Sufficient Condition** Let $f'\ x = f''\ x = \cdots = f\ n;1\ x = 0$, but $fn(x*) \neq 0$, Then $f\ (x)$ is:
- A minimum value of $f\ (x)$ if $f^n\ x* > 0$ and $n$ is even
- A maximum value of $f\ (x)$ if $f^n\ x* < 0$ and $n$ is even
- Neither a maximum nor a minimum if $n$ is odd.

**Note:** When $n$ is odd, the point $x*$ is
neither a maximum nor a minimum. In this case the point x∗ is called a **point of inflection**



**Example.** Determine the maximum and minimum values of the function:

$f\ x = 2x^3 - 3x^2 - 12x + 4$

**Solution.**

$f\ x = 2x^3 - 3x^2 - 12x + 4$

$\bar{f}(x) = 6x^2 - 6x - 12 = 6\,(x+1)(x-2)\,,\ \bar{f}(x) = 0$

$at\ x = -1, x = 2$

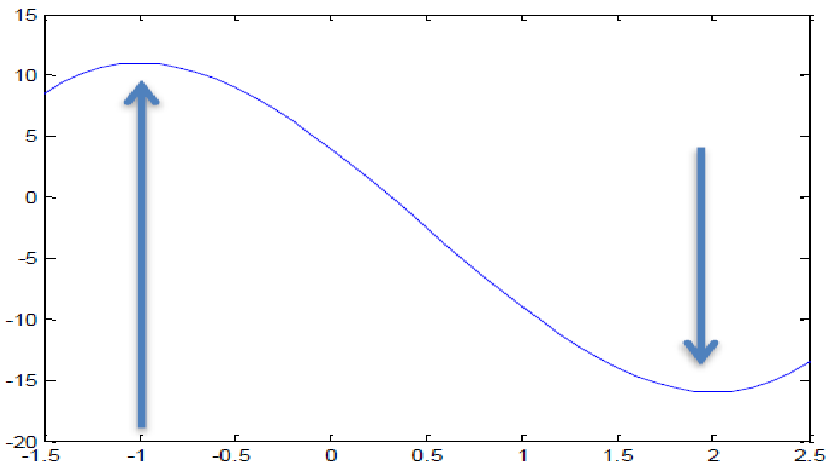
$\bar{\bar{f}} = 12x - 6 = 6\,(2x - 1) = 0$

At $x = -1$, $f'' = -18$, and hence $x = -1$ is a relative maximum.

$fmax(x) = f\ (-1) = 11$

At $x = 2$, $f'' = 18$, and hence $x = 2$ is a relative minimum.

$fmin\ (x) = f\ (2) = -16$




**Example.** Determine the maximum and minimum values of the function:

$f(\,x) = 12x^{\,5} - 45x^4 + 40x^{\,3} + 5$

**Solution.**

$f'(x) = 60x^{\,4} - 180x^{\,3} + 120x^{\,2} = 60x^{\,2}\,(x-1)(\,x-2)\,,\ f\ x = 0\ at\ x = 0, x = 1, x = 2$

$f'' = 60(\,4x^{\,3} - 9x^{\,2} + 4x)$

At $x = 1$, $f'' = -60$, and hence $x = 1$ is a relative maximum.

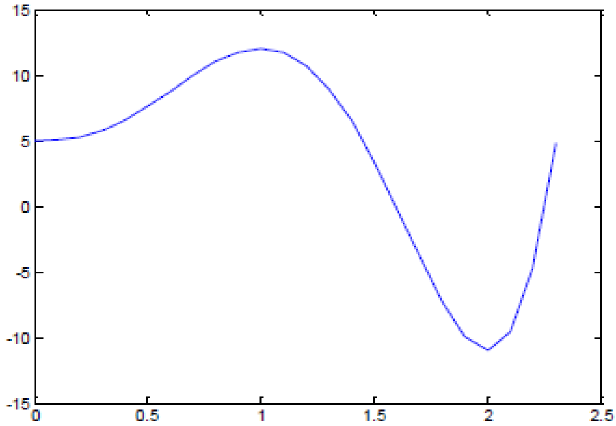
$fmax(x) = f\ (1) = 12$

At $x = 2$, $f'' = 240$, and hence $x = 2$ is a relative minimum.

$fmin(x) = f\ (2) = -11$

At $x = 0$, $f'' = 0$, and hence we must investigate the next derivative

$f''' = 60\ (12x^2 - 18x + 4)$ , at x(0) , $f''' = 240$




Since $f'''\ x \neq 0$, $at\ x = 0$, $x = 0$ is neither a maximum nor a minimum, and it is an **inflection point**

## Multivariable without Constraints

• The following two theorems:

**Necessary Condition** If $f(X)$ has an extreme point (maximum or minimum) at $X = X*$ and if the first partial derivatives of $f(X)$ exist at $X*$, then:

$$\frac{\partial f}{\partial x1}(\mathrm{x}\ *) = \frac{\partial f}{\partial x2}(\mathrm{x}\ *) = \cdots = \frac{\partial f}{\partial xn}(\mathrm{x}\ *) = 0$$

The following two theorems:

**Sufficient Condition** A sufficient condition for a stationary point $X*$ to be an extreme point is that the matrix of second partial derivatives (**Hessian matrix**) of $f(X)$ evaluated at $X*$ is:

- ❖ Positive definite when $X*$ is a relative minimum point.
- ❖ Negative definite when $X*$ is a relative maximum point.

The Hessian matrix of $f(X)$ evaluated at $(x1*, x2*)$ is:

$$j = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial {x^2}_1} & \frac{\partial^2 f(x)}{\partial \mathrm{x1}\, \partial \mathrm{x2}} \\ \frac{\partial^2 f(x)}{\partial \mathrm{x1}\, \partial \mathrm{x2}} & \frac{\partial^2 f(x)}{\partial {x^2}_2} \end{bmatrix}$$

- Note:

✓ A matrix $A$ of order of $n$ will be positive definite if and only if all the values $A1$, $A2$, …, $An$ (determinant) are positive.

✓ A matrix $A$ of order of $n$ will be negative definite if and only if the sign of $Aj$ is $(-1)^j$ $for\ j$ = 1, 2, . . . , $n$

A1= $[a_{11}]$, A2=$\begin{bmatrix} a11 & \mathrm{a12} \\ \mathrm{a21} & \mathrm{a22} \end{bmatrix}$, $An = \begin{bmatrix} a11 & a12 & \dots & a1n \\ a21 & a22 & \dots & a2n \\ an1 & an2 & \dots & ann \end{bmatrix}$

Note:

✓ **Saddle Point.** In the case of a function of two variables, $f$ $(x, y)$, the Hessian matrix may be neither positive nor negative definite at a point $(x_*, y_*)$ at which:

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} = 0$, in such a case, the point $(x_*, y_*)$ is called a saddle point.

✓ As an example, consider the function $f\ (x, y) = x2 - y2$. For this function:

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = -2y$, x* = 0, y* = 0

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f(x)}{\partial \mathrm{xy}} = 0, \quad \frac{\partial^2 f(x)}{\partial \mathrm{yx}} = 0$$

The Hessian matrix of $f$ is given by: $j = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial {x^2}_1} & \frac{\partial^2 f(x)}{\partial \mathrm{x1}\, \partial \mathrm{x2}} \\ \frac{\partial^2 f(x)}{\partial \mathrm{x1}\, \partial \mathrm{x2}} & \frac{\partial^2 f(x)}{\partial {x^2}_2} \end{bmatrix}$

$j = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, $j_1$ =2, $j_2$=-4, Aj is $j^{(-1)}$

Since this matrix is neither positive definite nor negative definite, the point $(x_* = 0, y_* = 0)$ **is a saddle point.**

**Example.** Find the extreme points of the function

$f\,(x1, x2) = 20x1 + 26x2 + 4x1x2 - 4x1^2 - 3x2^2$, the Hessian matrix of $f$, find the nature of these extreme points?

**Solution. The necessary conditions for the existence of an extreme point are:**

$\frac{\partial f}{\partial x1} = 20 + 4x2 - 8x1 = 0,\ \boldsymbol{x1} = \frac{\mathbf{20+4}\boldsymbol{x}\mathbf{2}}{\mathbf{8}}$ , $\frac{\partial f}{\partial x2} = 26 + 4x1 - 6x2 = 0,$

$\boldsymbol{x1} = \frac{\mathbf{6}\boldsymbol{x}\mathbf{2-26}}{\mathbf{4}}$

$\frac{\mathbf{20+4}\boldsymbol{x}\mathbf{2}}{\mathbf{8}} = \frac{\mathbf{6}\boldsymbol{x}\mathbf{2-26}}{\mathbf{4}}$ $\leftrightarrow$ 10 + 2x2 = 6x2 -26 $\leftrightarrow$ x2 =9, x1 = 7

The extreme points of the function: $x1 = 7$, $x2 = 9$.

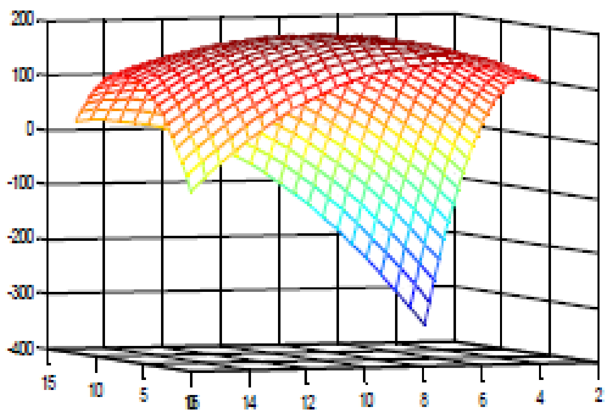
To find the Hessian matrix, we have to find the second-order partial derivatives of, $f$ (x1, x2):

$\frac{\partial^2 f(x)}{\partial {x^2}_1} = -8, \frac{\partial^2 f(x)}{\partial {x^2}_2} = -6 \quad \frac{\partial^2 f(x)}{\partial \mathrm{x1}\,\partial \mathrm{x2}} = 4, \frac{\partial^2 f(x)}{\partial \mathrm{x1}\,\partial \mathrm{x2}} = 4$

The Hessian matrix of $f$ is given by: $\mathrm{j} = \begin{bmatrix} -8 & 4 \\ 4 & -6 \end{bmatrix}$, The extreme points of the function : $x1 = 7$, $x2 = 9$.

$\mathrm{J}_1 = [-8]$, $\mathrm{j}_2 = \left[\begin{bmatrix} -8 & 4 \\ 4 & -6 \end{bmatrix}\right] = 32$

| Point($\boldsymbol{x1}, \boldsymbol{x2}$) | Value $J1$ | Value $J2$ | Nature of ($\boldsymbol{J1}, \boldsymbol{J2}$) | Nature of ($\boldsymbol{x1}, \boldsymbol{x2}$) | $\boldsymbol{f}$ ($\boldsymbol{x1}, \boldsymbol{x2}$) |
|---|---|---|---|---|---|
| (7, 9) | -8 | +32 | Negative definite | Relative maximum | 187 |



**Example.** Find the extreme points of the function

$f\,(x1, x2) = x1^3 + x2^3 + 2x1^2 + 4x2^2 + 6$

, the Hessian matrix of $f$, find the nature of these extreme points?

**Solution.** The necessary conditions for the existence of an extreme point are:

$$\frac{\partial f}{\partial x1} = 3x1^2 + 4x1 \leftrightarrow x1(3x1 + 4) = 0\ ,\ x1\ =\ 0, x1\ = -\frac{4}{3}$$

$$\frac{\partial f}{\partial x2} = 3x2^2 + 8x2 \leftrightarrow x2(3x2 + 8) = 0, x2 = 0, x2 = -\frac{8}{3}$$

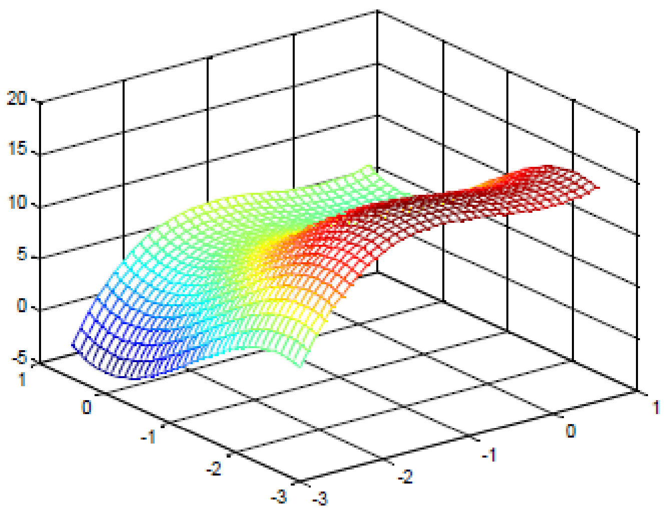
Extreme points of the function are the points: (0, 0),(0,-$\frac{8}{3}$), (-$\frac{4}{3}$, 0), $(-\frac{4}{3}, -\frac{4}{3})$ To find the Hessian matrix, we have to find the second-order partial derivatives of f

$$\frac{\partial^2 f(x)}{\partial x^2{}_1} = 6x1 + 4, \frac{\partial^2 f(x)}{\partial x^2{}_2} = 6x2 + 8\quad \frac{\partial^2 f(x)}{\partial x1\, \partial x2} = 0, \frac{\partial^2 f(x)}{\partial x1\, \partial x2} = 0$$

The Hessian matrix of $f$ is given by: $j = \begin{bmatrix} 6x1 + 4 & 0 \\ 0 & 6x2 + 8 \end{bmatrix}$, The extreme points of the function : (0, 0),(0,-$\frac{8}{3}$), (-$\frac{4}{3}$, 0), $(-\frac{4}{3}, -\frac{4}{3})$

$J_1 = [6x1 + 4]$, $j_2 = \left[\begin{bmatrix} 6x1 + 4 & 0 \\ 0 & 6x2 + 8 \end{bmatrix}\right]$

| Point($\boldsymbol{x1}, \boldsymbol{x2}$) | Value$J1$ | Value$J2$ | Nature of($\boldsymbol{J1}, \boldsymbol{J2}$) | Nature of($\boldsymbol{x1}, \boldsymbol{x2}$) | $\boldsymbol{f}\,(\boldsymbol{x1}, \boldsymbol{x2})$ |
|---|---|---|---|---|---|
| (0, 0) | 4 | +32 | Positive definite | Relative minimum | 6 |
| (0,-8/3) | 4 | -32 | indefinite | Saddle point | 418/27 |
| (-4/3,0) | -4 | -32 | indefinite | Saddle point | 418/27 |
| | -4 | +32 | Negative definite | Relative maximum | 50/3 |



**Multivariable with Constraints (equality constraints)**

• **Solution by Direct Substitution**

✓ Constraints are substituted into the original objective function

✓ The new objective function is not subjected to any constraint,

✓ The optimum can be found by using the unconstrained (single or multivariable) optimization techniques

**Example.** Find the extreme points of the function

$f\,(x1, x2, x3)= x1^2 + (x2+1)^2 + (x3-1)^2$

the Hessian matrix of $f$, find the nature of these extreme points, the function subject to the following constraint $x1 + 5x2 - 3x3 = 6$

**Solution.** Direct Substitution $x_1 + 5x_2 - 3x_3 = 6 \leftrightarrow x3 = \frac{5x2+x1-6}{3}$

$f\,(x1, x2)= x1^2 + (x2+1)^2 + (\frac{5x2+x1-6}{3} - 1)^2$

The new objective function is not subjected to any constraint

$f\,(x1, x2)= x1^2 + (x2+1)^2 + \frac{1}{9}(5x2 + x1 - 9)^2$

$\frac{\partial f}{\partial x1} = 2x1 + \frac{2}{9}(5x_2 + x_1 - 9) = 0 \leftrightarrow 20x_1 + 10x_2 - 18 = 0 \leftrightarrow x1 = \frac{18-10x2}{20}$

$\frac{\partial f}{\partial x2} = 2(x2+1) + \frac{10}{9}(5x_2 + x_1 - 9) = 0 \leftrightarrow 10x_1 + 68x_2 - 72 = 0 \leftrightarrow x2 = \frac{72-68x2}{10}$

$\frac{18-10x2}{20} = \frac{72-68x2}{10} \leftrightarrow x2 = \frac{72-9}{68-5} = \frac{63}{63} = 1 \leftrightarrow x1 = \frac{18-10(1)}{20} = \frac{8}{20} = 0.4 \leftrightarrow$

$x3 = \frac{x1+5x2-6}{3} = -0.2$ **(x1 =0.4, x2 =1, x3 = -0.2 extreme points)**

To find the Hessian matrix, we have to find the second-order partial derivatives of

$\frac{\partial f}{\partial x1} = 2x1 + \frac{2}{9}(5x_2 + x_1 - 9) \leftrightarrow \frac{\partial^2 f(x)}{\partial x^2{}_1} = \frac{20}{9}$, $\qquad \frac{\partial^2 f(x)}{\partial x1\,\partial x2} = \frac{10}{9}$,

$\frac{\partial f}{\partial x2} = 2(x2+1) + \frac{10}{9}(5x_2 + x_1 - 9) = 0$, $\quad \frac{\partial^2 f(x)}{\partial x^2{}_2} = \frac{68}{9}$, $\qquad \frac{\partial^2 f(x)}{\partial x2\,\partial x1} = \frac{10}{9}$,

The Hessian matrix of $f$ is given by: $j = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x^2{}_1} & \frac{\partial^2 f(x)}{\partial x1\,\partial x2} \\ \frac{\partial^2 f(x)}{\partial x2\,\partial x1} & \frac{\partial^2 f(x)}{\partial x^2{}_2} \end{bmatrix}$ $j = \begin{bmatrix} \frac{20}{9} & \frac{10}{9} \\ \frac{10}{9} & \frac{68}{9} \end{bmatrix}$

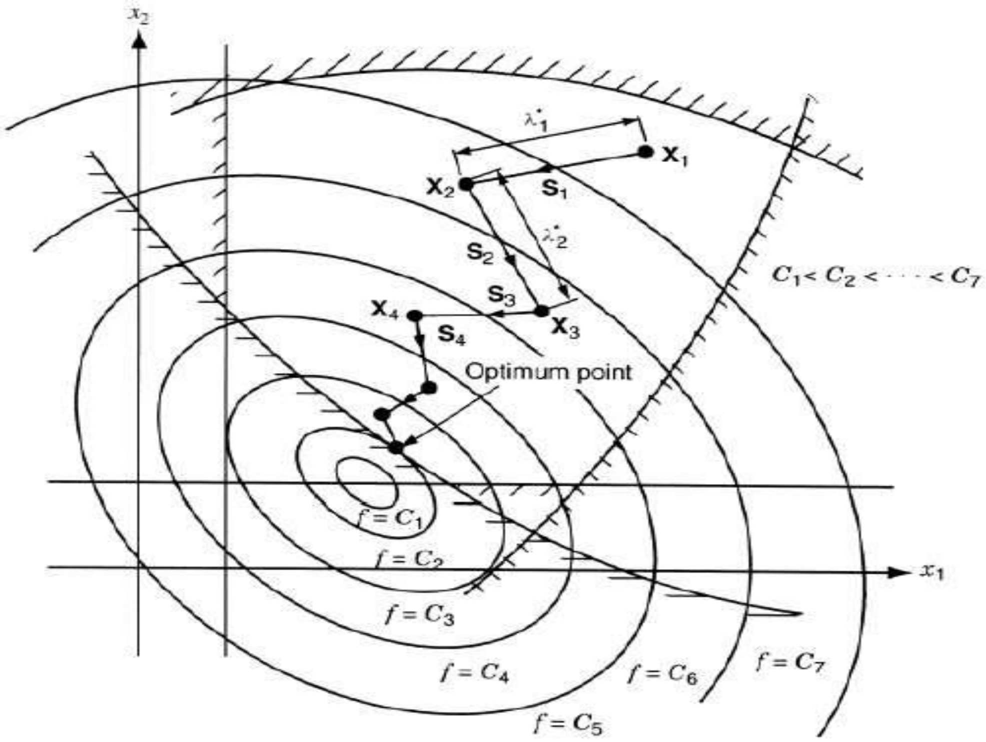
$J_1 = \left[\frac{20}{9}\right]$, $j_2 = \begin{bmatrix} \frac{20}{9} & \frac{10}{9} \\ \frac{10}{9} & \frac{68}{9} \end{bmatrix}$

| Point($x1$, $x2$) | Value$J1$ | Value$J2$ | Nature of($J1$, $J2$) | Nature of($x1$, $x2$) | $f\,(x1, x2)$ |
|---|---|---|---|---|---|
| (0.4,1, -0.2) | +2.2 | +140 | Positive definite | Relative minimum | 5.6 |

# Numerical Optimization Techniques

The basic philosophy of most of the numerical methods of optimization is to produce a sequence of improved approximations to the optimum according to the following scheme:

- Start with an initial trial point $X1$.
- Find a suitable direction $Si$ ($i = 1$ to start with) that points in the general direction of the optimum.
- Find an appropriate step length $\lambda i*$ for movement along the direction $S_i$.
- Obtain the new approximation $Xi+1$ as: $Xi+1 = Xi + \lambda i *$
- If $Xi+1$ is optimum, stop the procedure. Otherwise, set a new $i = i + 1$ and repeat step (2) onward.




- The iterative procedure is valid for unconstrained as well as constrained optimization problems.
- The efficiency of an optimization method depends on the efficiency with which the quantities $\lambda i *$ and $Si$ are determined.

- If $f\ (X)$ is the objective function to be minimized, the problem of determining $\lambda i *$ reduces to finding the value $\lambda i = \lambda i *$ that minimizes $f\ (Xi{+}1) = f\ (Xi + \lambda i\ Si) = f(\lambda i\ )$ for fixed values of $Xi$ and $Si$ .
- Since $f$ becomes a function of one variable $\lambda i$ only, the methods of finding $\lambda i$ are called **one-dimensional minimization methods.**

• A Example of one-dimensional minimization methods:

a) Elimination Methods (Fibonacci Method & Golden Section Method)

b) Interpolation Methods (Newton Method)

# Fibonacci Method

• Fibonacci method can be used to find the minimum of a function even if the function is no differentiable and not continuous.

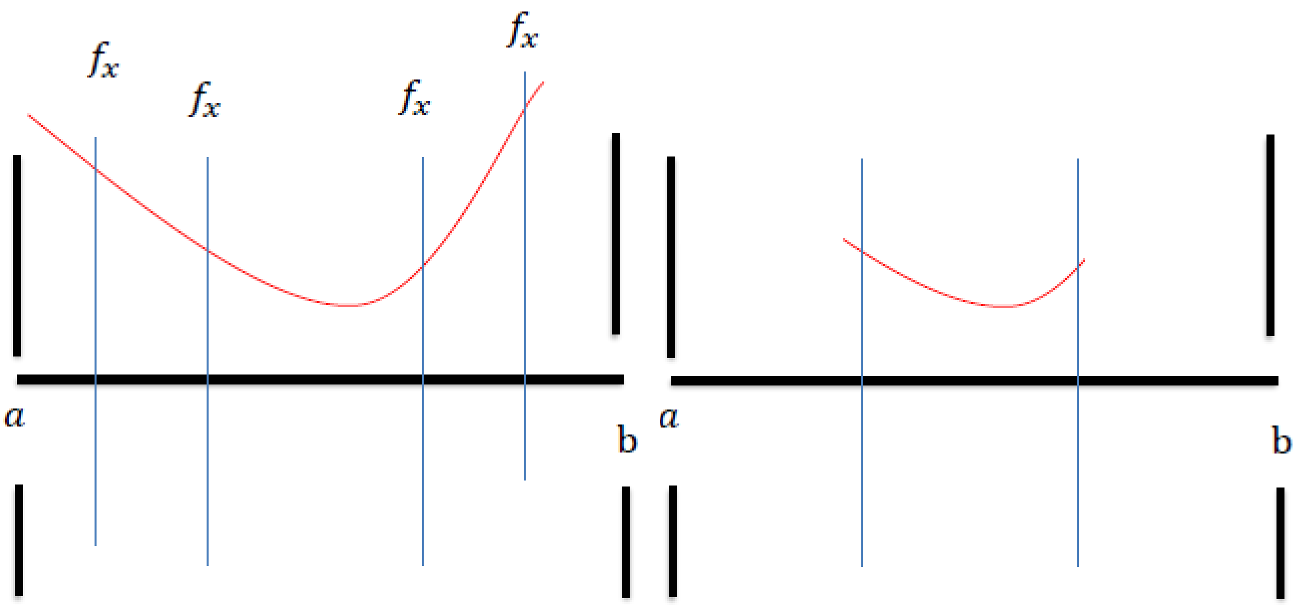
• Elimination Methods

• This method makes use of the sequence of Fibonacci numbers,$\{F_n\}$ for placing the experiments.

• These numbers are defined as: $F_0 = F_1 = 1, Fn = F_{n-1} + F_{n-2}$ which yield the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . . .

• The method is based on evaluating the objective function at different points in the interval.

• Narrow the range until the minimizer is “boxed in” with sufficient accuracy.

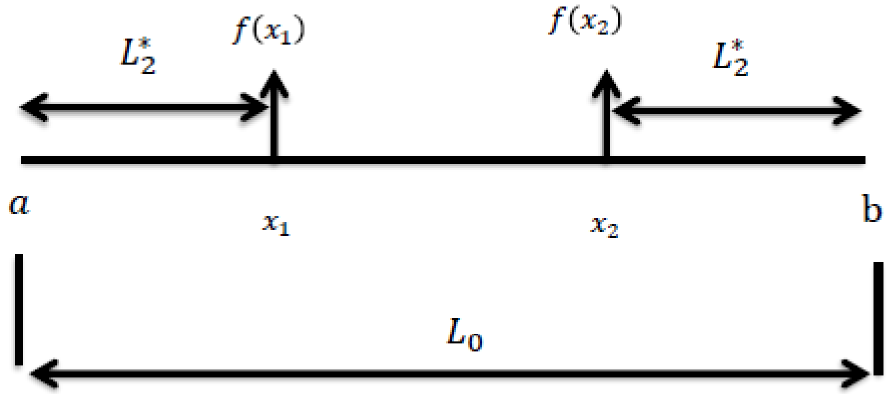


- The initial interval of uncertainty, by $a \le x \le b$ , has to be known.
- The function being optimized has to be unimodal.
- The total number of experiments $n$ has to be determined.

# Procedure

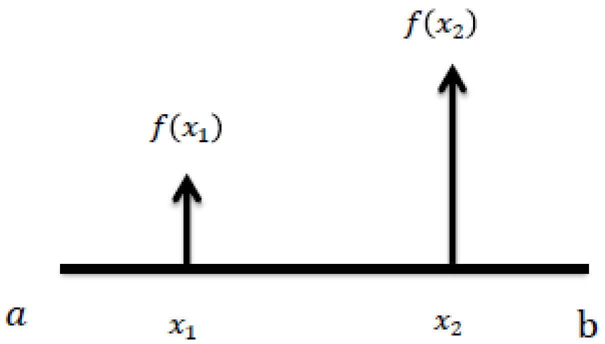
- Step 1: Define $L_0 = b - a$ as the initial interval of uncertainty.
- Step 2: Define: $L_2{*}=\frac{\mathrm{Fn-2}}{\mathrm{Fn}}\mathrm{L}_0$ and place the first two experiments at

points $x1$ and $x2$. This gives: $x1 = a + L_2{*}$, $x2 = = \mathrm{b} + L_2{*}$
*
- Step 3: Find $f\ (x1)$ and $f(x2)$




- Step 4: Compare $f\ (x1)$ and $f(x2)$

✓ If $f\ (x1) < f\ (x2)$

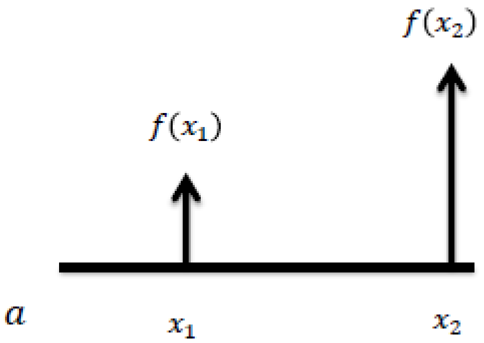



- Step 4: Compare $f\ (x_1)$ and $f(x_2)$

✓ If $f\ (x_1) < f\ (x_2)$

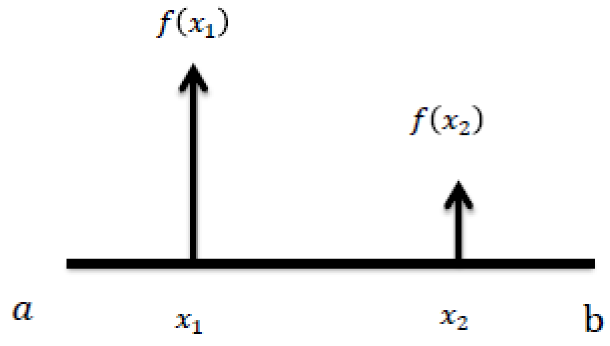
❖ Delete $x_2$, $b$

❖ $x_j = a + (x_2 - x_1)$



- Step 4: Compare $f\ (x1)$ and $f(x2)$

✓ If $f\ (x1) > f\ (x2)$

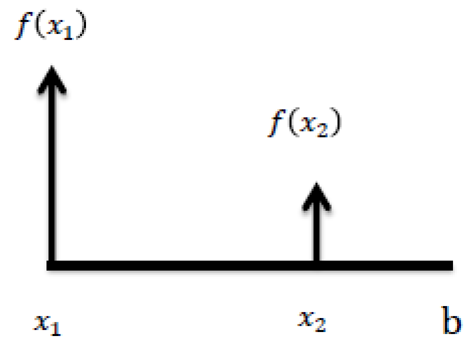



Step 4: Compare $f\ (x1)$ and $f(x2)$

✓ If $f\ (x1) > f\ (x2)$

Delete [a,x1]

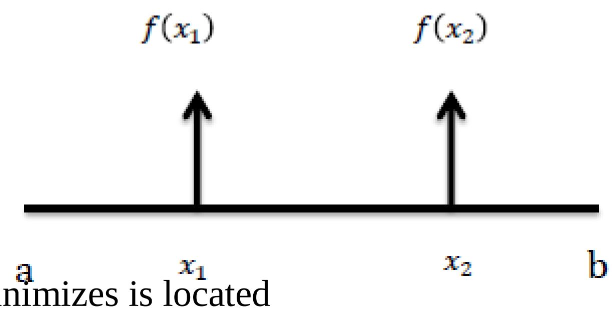
❖ $x_j = \mathrm{x1} + (\mathrm{b} - x2)$




- Step 4: Compare $f\ x1$ and $f(x2)$

✓ If $f\ (x1) = f(x2)$

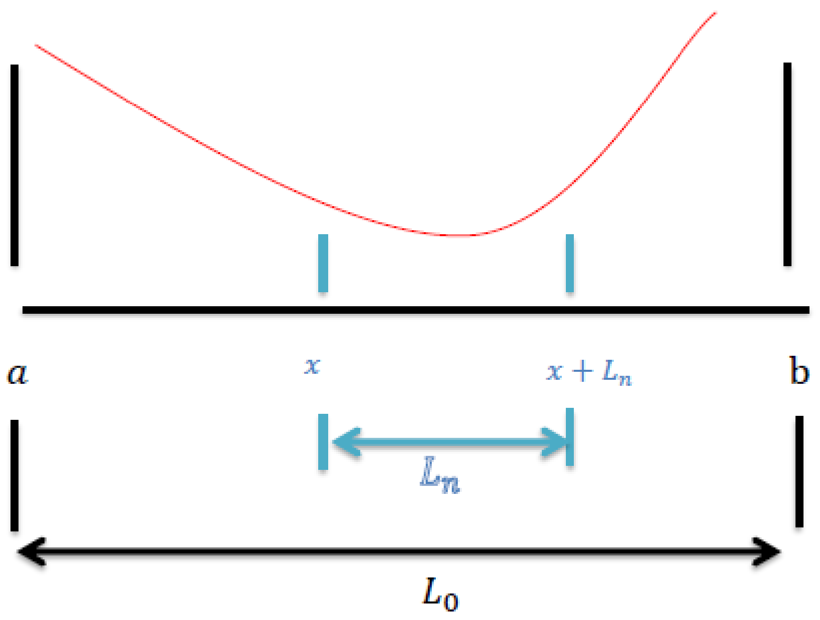
❖ $x_j = a_j + \frac{\mathrm{Fn-j}}{\mathrm{Fn}}(b - a)_j$




Step 5: After n experiments

✓ Find the final interval $[x, x + L_n]$ that minimizes is located

✓ Calculate the ratio of the final to the initial interval of uncertainty: $\frac{Ln}{L0}$



**Example.** Minimize the following function in the interval [-1,1] by using Fibonacci method
using n = 4?

$f(x) = e^x - \sin x$

**Solution.**

$n = 4, a = -1, b = 1, L_0 = b - a = 1 - -1 = 2,$