2- interval halving method (bisection method)

The interval halving method required the signal of the function:

The following steps are used:

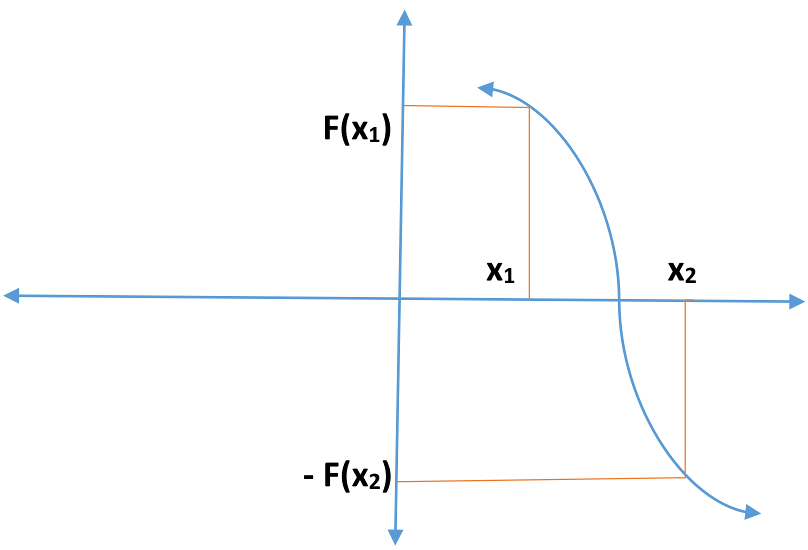
1. Bracket the solution by finding two values of x

one where f(x)<0 named x1

another where the function f(x)>0 named x2

2. Evaluate the function f(x)at the mid-point of the bracket $x_3=\frac{x1+x2}{2}$
3. Replace the value of the variables x1 or x2 with x3 depending on the sign of the functions if f(x1) and f(x3) have the same sign x1=x3, x2=x2. If f(x2) and f(x3) have the same sign x2=x3, x1=x1
4. Repeat above calculation from step 2 until the value of x3 is still constant a reach required error value.

Error req.= $\left|\frac{x3new-x3old}{x3old}\right|$



$x_3=\frac{x1+x2}{2}$

Example: find the root of the following equation using interval halving method (bisection method then range (0.2<x<0.9)

$F(x)=e^{-x}-x$ 4 decimal place

Solution:

X1	X2	X3	F(x1)=e^{-x1}-x	F(x3) =e^{-x3}-x	Erorr
0.2	0.9	0.55	0.618731	-0.49343	------
0.55	0.9	0.725	0.02695	-0.49343	0.318182
0.55	0.725	0.6375	0.02695	-0.24068	0.12069
0.55	0.6375	0.59375	0.02695	-0.10889	0.068627
0.55	0.59375	0.571875	0.02695	-0.0415	0.036842
0.55	0.571875	0.560938	0.02695	-0.00741	0.019126
0.560938	0.571875	0.566406	0.009736	-0.00741	0.009749
0.566406	0.571875	0.569141	0.001155	-0.00741	0.004828
0.566406	0.569141	0.567773	0.001155	-0.00313	0.002402
0.566406	0.567773	0.56709	0.001155	-0.00099	0.001204
0.56709	0.567773	0.567432	8.38E-05	-0.00099	0.000603
0.56709	0.567432	0.567261	8.38E-05	-0.00045	0.000301
0.56709	0.567261	0.567175	8.38E-05	-0.00018	0.000151
0.56709	0.567175	0.567133	8.38E-05	-5E-05	7.53E-05
0.567133	0.567175	0.567154	1.68E-05	-5E-05	3.77E-05
0.567133	0.567154	0.567143	1.68E-05	-1.7E-05	1.88E-05
0.567143	0.567154	0.567149	6.41E-08	-1.7E-05	9.42E-06
0.567143	0.567149	0.567146	6.41E-08	-8.3E-06	4.71E-06
0.567143	0.567146	0.567145	6.41E-08	-4.1E-06	2.35E-06

	A	B	C	D	E	F
1	X1	X2	X3	F(x1)	F(x3)	Error
2	0.2	0.9	=(A2+B2)/2	=exp(-A2)-A2	= exp(-C2)-C2	
3	=if(D2*E2>0;C2;A2)	=if(D2*E2>0;B2;C2)				=abs((c3-c2)/c3)
4	نسحب	نسحب	نسحب	نسحب	نسحب	نسحب
5						

Example2: solve x^3-9x+1=0 for the root between x1=2 and x2=4 by bisection method.

Solution: given f(x)= x^3-9x+1

∴ f(2)=-9, f(4)=29 the root between 2 and 4

$X_3 = \frac{2+4}{2} = 3$

$f(x_3)$ =f(3)=1

$f(x_1)$ =f(2)=-9

$f(x_2) = f(4) = 29$

$x_3 = x_2 = 3$

$x_1 = x_1 = 2$

$X_3 = \frac{3+2}{2} = 2.5$

$f(x_3) = f(2.5) = (2.5)^3 - 9(2.5) + 1 = -23.375$

x1	x2	x3	f(x1)	f(x2)	f(x3)	error
2	4	3	-9	29	1	
2	3	2.5	-9	1	-5.875	0.166667
2.5	3	2.75	-5.875	1	-2.95313	0.1
2.75	3	2.875	-2.95313	1	-1.11133	0.045455
2.875	3	2.9375	-1.11133	1	-0.09009	0.021739
2.9375	3	2.96875	-0.09009	1	0.446259	0.010638
2.9375	2.96875	2.953125	-0.09009	0.446259	0.175922	0.005263
2.9375	2.953125	2.945313	-0.09009	0.175922	0.042378	0.002646
2.9375	2.945313	2.941406	-0.09009	0.042378	-0.02399	0.001326

H.W using interval halving method to find the root of the :

$f(x) = x^3 - 9x + 1$ at ($x1=0$ $x2=1$) and($x1=1$ $x2=2$)

3- Newton-Raphson method

the algorithm of the Newton-Raphson method is obtained from a Taylor-series expansion of f(x) approximation to a root

$$f(x) = f(x0) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!} + f''''(x) + \cdots$$

(Taylor series)

$h = x_1 - x_0$, $f(x) = 0$ 0 0 0

the basic formula for the iteration in this method is :

$$x_1 = x_0 - \frac{f(x0)}{f'(x0)}$$

Example: find the real root for the following equation using Newton – Raphson method

$f(x) = \log(x) - 1 + 1/x2 = 0$;let $x_o = 10$

$\log(x) = \frac{ln(x)}{ln10} = \frac{lnx}{2.303}$

$\bar{f}(x) = \frac{1}{2.303} * \frac{1}{x} - \frac{2}{x^3}$

X_0	$f(x0)$	$\bar{f}(x0)$	X_1	Erorr
10	0.01	0.0414	9.7586	0.0241
9.7586	-0.001	0.0424	9.7612	0.003
9.7612	0.00000	0.0423	9.7612	0.0000

Root =9.7612

	A	B	C	D	E
1	X_0	$F(x_0)$	$dF(x_0)$	x_n	Error
2	10	=log(A_2) - 1+1/A2^2	=1/2.303*1/A_2-2/A_2^3	=A2-B2/C2	=abs((D_2-A_2)/A_2)
3	=D_2				
4	نسحب	نسحب	نسحب	نسحب	نسحب
5					

Example : by Newton-Raphson method find the root of $y=x^7-100$

$X_0=3$

Solution :

$\bar{y} = 7x^6$

$x_1 = x_0 - \frac{f(x0)}{\bar{f}(x0)}$

$x_1 = 3 - \frac{x7-100}{7x6} = 3 - \frac{1187}{5103}$

$x_1=2.7673$

$x_1 = 2.7673 - \frac{243.06137}{3144.233}$

$x_1=2.69$

$x_1 = 2.69 - \frac{19.453}{2652.233}$

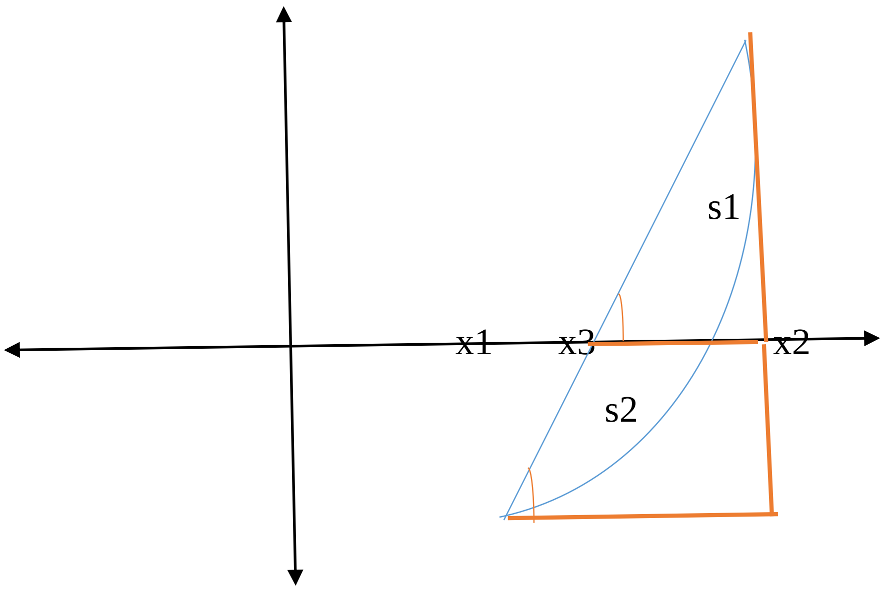
$x_1=2.6826$

secant method

in this method it is required to estimate the function f(x)=0 at the point (to value of x, one of them greater that root (b) and another is lass then root (a), where R is the root

x1=a, a<R

x2=b, b>R



$$\left(\frac{dy}{dx}\right)s1=\left(\frac{dy}{dx}\right)s2$$

$$\frac{f(x2)-0}{x2-x3}=\frac{f(x2)-f(x1)}{x2-x1}$$

$$x_3 = x_2 - \frac{x2-x1}{f(x2)-f(x1)}.f(x2)$$

continue until

$(x_{3new} - x_{3old}) / x_{3old} < E$

Check f(x1),f(x2),f(x3)

If sign f(x1) =sign f(x3)

x1=x3

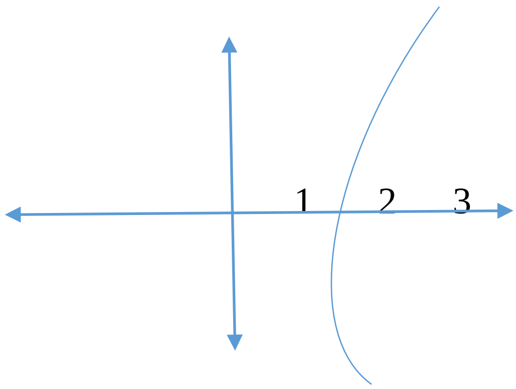
x2=x2

if sign f(x2) =sign f(x3)

x2=x3

x1=x1

Example: find the real root of the following equation using secat method $f(x) = x^3 - 5x^2 - 2x + 10 = 0$ In (1,3) 5decimal place and error E=0.0009 .



x1	x2	f(x1)	f(x2)	x3	f(x3)	Error
1	3	4	-14	1.444444	-0.30727	
1	1.444444	4	-0.30727	1.412739	0.014955	0.02195
1.412739	1.444444	0.014955	-0.30727	1.41421	3.24E-05	0.001042
1.41421	1.444444	3.24E-05	-0.30727	1.414214	7.01E-08	2.26E-06
1.414214	1.444444	7.01E-08	-0.30727	1.414214	1.52E-10	4.88E-09
1.414214	1.444444	1.52E-10	-0.30727	1.414214	3.29E-13	1.05E-11
1.414214	1.444444	3.29E-13	-0.30727	1.414214	0	2.29E-14

	A	**B**	**C**	**D**	**E**	**F**
1	**X1**	**X2**	**F(x1)**	**F(x2)**	**x3**	**F(x3)**
2	**1**	**3**	**=A2^3 - 5*A2^2 - 2*A2+10**	**=B2^3 - 5*B2^2 - 2*B2+10**	**=B2-((B2-A2)/(D2-C2)*D2**	**=E2^3 - 5*E2^2 - 2*E2+10**
3	**=if(C2*F2>0; E2;A2)**	**=if(D2*F2>0 ;E2;B2)**				
4	**نسحب**	**نسحب**	**نسحب**	**نسحب**	**نسحب**	**نسحب**
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H.W

1. find the root of equation using secant method

$F(x) = e^x - 3x^2$ in (3,4) 5deci. Pla.

2. find the real root of the following equation using Newton-Raphson method

$F(x) = x - \frac{x3-3x2-1}{3x2-1}$ initial guess=-1

3. a process furnace heating 150 mol/hr of vapor phase ammonia the rate of heat addition to the furnace is 10^5J/hr . the ammonia feed temperature is 500 k (T1) assume ideal gas and using the following equation for heat capacity at constant pressure?

$cp = a + bT + cT^2 + DT^2$

$Q = \Delta\bar{H} = ncpdT = n[aT + \frac{b}{2}.T^2 + \frac{c}{3}.T^3 - DT^{-1}]$

$Q = \Delta\bar{H} = n[a(T_2 - T_1) + \frac{b}{2}(T_2^2 - T_1^2) + \frac{c}{3}(T_2^3 - T_1^3) - D(T_2^{-1} - T_1^{-1})]$

Solve this equation numerically for T_2 use Newton-Raphson

$a = 3.57\frac{J}{mol.k}$, $f(T_2) = 0$

$b = 2.02*10^{-3}$, $c = 0$, $D = -0.186*10^5$

Example:

$(p - \frac{a}{v2})(v - b) = RT$

P=50

T=-100°C = $173K^0$

A = 1.33,   b=0.366,   R=0.082

Solution:

$V_0 = \frac{RT}{p} = \frac{0.082*173}{50} = 0.28372$

$V_n = V_0 - \frac{f(v)}{\bar{f}(v)}$

V_0	$f(v)$	$\bar{f}(v)$	V_n
0.28372	-19.65946	76.10564	0.54204
0.54204	-4.58717	51.58645	0.63096
0.63201	-0.05279	50.53998	0.63201
0.63201	-0.00000	50.32682	0.63201

Root = 0.63201

Array

A. One dimensional array (vector)

Raw vector A= [1 2 3 4 5]

Column vector B = $\begin{matrix} 1 \\ 6 \\ 8 \end{matrix}$

B. Two-dimension array (matrix)

C = $\begin{matrix} 1 & 5 & 0 \\ 3 & 7 & 4 \end{matrix}$ D = $\begin{matrix} 4 & 8 \\ 0 & 2 \\ 1 & 1 \end{matrix}$

C. Three-dimension array (matrix)

A= $\begin{matrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 6 & 4 \end{matrix}$

Array (vectors and matrix)

A matrix is rectangular or square array of elements I which both value and its position of element the size is described by it numbers of raw and column. for example, a matrix A(m,n) it said to be of size (n,m).A matrix is generally represented by enclosing its elements a pair of square brackets ([ ]) .suppose matrix A(n,m)

A= $\begin{matrix} a11 & a12 \dots \dots & a1m \\ a21 & a22 \dots \dots & a2m \\ an1 & an2 \dots \dots & anm \end{matrix}$

Where A:is the matrix name

n: no. of rows

m: no. of columns

a matrix with only one raw (1,n) in size is called a raw victor (or raw matrix)

Ex: A = [1 3 10 15]

Sol: A = (1,m) = (1,4)

A matrix with only one column (n,1) in size is called a column vector or (column matrix)

$$\text{Ex: } A = \begin{matrix} 2 \\ 4 \\ 8 \end{matrix} = A(n,1)$$

A matrix called square matrix if its number of rows is equal to its number of column.

$$\text{Ex: } A = \begin{matrix} 2 & 4 & 6 \\ 5 & 10 & 15 \\ 10 & 20 & 30 \end{matrix} \quad A(m,m) = A(n,n), \quad n=3,\ m=3$$

Upper triangle

$$\text{If } A = \begin{matrix} 2 & 4 & 6 \\ 5 & 1 & 5 \\ 9 & 4 & 3 \end{matrix}$$

second diagonal   main diagonal

lower triangle

if each of the main diagonal elements is one it is said to be identity or unit matrix.

$$\text{Ex: } B = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix},\ B(3,3)$$

- Upper triangle matrix

$$C = \begin{matrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 8 & 5 & 1 \end{matrix}$$

- Lower triangle matrix

$$D = \begin{matrix} 1 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{matrix}$$

- Ones matrix
- $E = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$
- Zero matrix

$$F = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Determinant matrix

Suppose we heve matrix with size A(3,3)

$$A = \begin{matrix} 1 & 2 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{matrix} \longrightarrow A = \begin{matrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{matrix}$$

a_{11} minor matrix $\begin{matrix} a22 & a23 \\ a32 & a33 \end{matrix}$

a12 minor matrix $\begin{matrix} a21 & a23 \\ a31 & a33 \end{matrix}$

1) $A = \begin{matrix} 2 & 1 \\ 4 & 6 \end{matrix}$ =A(2,2) determinant = 12-4 =8

2) $A = \begin{matrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{matrix}$

d.A= $(-1)^{i+j} 2\begin{pmatrix} -1 & -2 \\ 3 & 1 \end{pmatrix} -1\begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix} +3\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$

= 2(-1+6) -1(3+4) +3(9+2) =36

3) $A = \begin{bmatrix} \overset{2}{1} & \overset{-4}{-2} & \overset{6}{3} & \overset{2}{1} \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Determinant of A equal:

R_2-2R_1
R_3+R_1

$$A = \begin{bmatrix} 1 & \overset{+}{-2} & \overset{-}{3} & \overset{+}{1} \\ 0 & 5 & -6 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\det.A = 1\left(5\begin{pmatrix}4 & -1\\2 & 1\end{pmatrix} - (-6)\begin{pmatrix}0 & -1\\1 & 1\end{pmatrix} + 0\begin{pmatrix}0 & 4\\-1 & 2\end{pmatrix}\right)$$

=1[5(4+2) +6(0+1) +0 (0+4)

=30+6

=36

Transpose of matrix (A^T):

The transpose of a matrix is a matrix that is obtained when rows are writing as column the symbol A^T is use to denoted transpose of A

Example: for a matrix D find D^T? D= [1 3 6]

Solution; $D^T = \begin{matrix}1\\3\\6\end{matrix}$

Example: for a matrix C find C^T?

$C = \begin{matrix}1\\3\\7\end{matrix}$

Solution: C^T =[1 3 7]

Example: for a matrix E find E^T?

$$E = \begin{matrix}2 & 5 & 1\\-2 & 4 & 0\\1 & 3 & 6\end{matrix}$$

Solution:

$$E^T = \begin{matrix}2 & -2 & 1\\5 & 4 & 3\\1 & 0 & 6\end{matrix}$$

Mathematical operation on matrix

a) Addition and subtraction

Two matrices of the same size can be added or subtracted

A =[aij], B =[bij], i=1,2,3,….n , j =1,2,3,…..m

Then their addition is amatrix of the same size

A +B = [aij] +[bij] = [cij]

And their substraction is a matrix of same size .

A-B = [aij] – [bij] = [dij]

Ex: $A = \begin{matrix} 5 & 8 & -4 \\ 2 & -4 & 10 \end{matrix}$, $B = \begin{matrix} 2 & 4 & -4 \\ 2 & -5 & 6 \end{matrix}$

Solution:

$C = A+B = [\begin{matrix} 5+2 & 8+4 & -4-4 \\ 2+2 & -4-5 & 10+6 \end{matrix}]$

$C = \begin{matrix} 7 & 12 & -8 \\ 4 & -9 & 16 \end{matrix}$

$D = A-B = [\begin{matrix} 5-2 & 8-4 & -4+4 \\ 2-2 & -4+5 & 10-6 \end{matrix}]$

$D = \begin{matrix} 3 & 4 & 0 \\ 0 & 1 & 4 \end{matrix}$

Q) find the addition and subtraction of A and B

A =[1 3 6 12] , B = [2 6 78 9 3] =Error.

A =[1 -1 -2] , $B = \begin{matrix} 2 \\ 6 \\ 7 \end{matrix}$ = Error.

$A = \begin{matrix} 5 \\ 6 \\ 7 \end{matrix}$, $B = \begin{matrix} 2 \\ 4 \\ 3 \end{matrix}$

$C = A + B = \begin{matrix} 7 \\ 10 \\ 10 \end{matrix}$

$D = A - B = \begin{matrix} 3 \\ 2 \\ 4 \end{matrix}$

b) Multiplication of matrices:

Multiplication two matrices A and B is possible if number of columns of matrix A is the equal to the number of rows of matrix B . the size of result matrix has size (rows of A , column of B)

Ex: A(3,2) * B(6,3) ⟶ error

: A(6,1) * B(2,3) ⟶ error

A(4,6) * B(6,2) ⟶ C(4,2)

A(3,3) * B(3,3) ⟶ C(3,3)

A(3,3) * B(3,1) ⟶ C(3,1)

A(1,3) * B(3,1) ⟶ C(1,1)

Example: Evaluate the products A*B for the following:

$$A = \begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 4 \end{matrix} \quad , \quad B = \begin{matrix} 2 & 1 \\ 4 & 5 \\ 1 & 7 \end{matrix}$$

$$C = \begin{bmatrix} 3*2+2*4+1*1 & 3*1+2*5+1*7 \\ 2*2+1*4+4*1 & 2*1+1*5+4*7 \end{bmatrix}$$

$$C = \begin{bmatrix} 15 & 20 \\ 12 & 35 \end{bmatrix}$$

c) Multiplication by scalar:

If a matrix is multiplied by a scalar k (number) the result matrix is obtain by multiply k by each element in matrix .

Ex: for matrix $A = \begin{bmatrix} 2 & 6 \\ 8 & -2 \end{bmatrix}$, find 2A

Solution: $C = 2A = \begin{bmatrix} 2*2 & 6*2 \\ 8*2 & -2*2 \end{bmatrix}$

$$C = \begin{bmatrix} 4 & 12 \\ 16 & -4 \end{bmatrix}$$

	A	B	C	D	E	F	G	H	I	J
1	mat	rix	A	mat	rix	B	A	+ **B**		
2	1	5	3	2	3	5	=(A_2:C_4) +	(D_2:F_4)		
3	4	-1	6	-1	5	7				
4	5	0	4	6	4	0	Ctrl+	shift +	enter	
5										
=mmulty	(A_2:C_4)									
; (D_2:F_4)										
Ctrl +shift	+enter						=MDETRM	(A_2:C_4)		

Or =minverse (A_2:C_4)/ =minverse (D_2:F_4)

Solution to set of linear algebraic equations:

The equation can be writing as;

$$\begin{cases} a11x1 \quad a12x2 \quad a13x3 \ldots \ldots a1nxn = b1 \\ a21x1 \quad a22x2 \quad a23x3 \ldots \ldots a2nxn = b2 \\ a31x1 \quad a32x2 \quad a33x3 \ldots \ldots a3nxn = b3 \\ \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots = \cdots \\ an1x1 \quad an2x2 \quad an3\,x3 \ldots \ldots annxn = bn \end{cases}$$

In which the coefficients (a_{ij}) and the right hand side coeffecints (bi) are known constant the above equation can be written as :

$$\begin{bmatrix} a11 & a12 & a13 \ldots \ldots & a1n \\ a21 & a22 & a23 \ldots \ldots & a2n \\ a31 & a32 & a33 \ldots \ldots & a3n \\ \ldots & \ldots & \ldots \quad \ldots \ldots & \ldots \\ an1 & an2 & an3 \ldots \ldots & ann \end{bmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_4 \end{vmatrix} = \begin{vmatrix} b1 \\ b2 \\ b3 \\ \ldots \\ bn \end{vmatrix}$$

Ex: for below system, write it in matrix form;

$3x_1 + 2x_2 + x_3 = 1$

$-x_2 + 3x_3 – x_1 = 0$

$4x_1 + 2x_3 = 2$

Solution;

$3x_1 + 2x_2 + x_3 = 1$

$-x_1 - x_2 + 3x_3 = 0$

$4x_1 + 0 + 2x_3 = 2$

$$\begin{bmatrix} 3 & 2 & 1 \\ -1 & -1 & 3 \\ 4 & 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & -1 & 3 \\ 4 & 0 & 2 \end{bmatrix} \quad , \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$A.x = b$

Solution of set of linear equations by

1. matrix inverse method.

$Ax = b \qquad x = \frac{b}{A} \quad , x = bA^{-1}$

$A^{-1} = \frac{adj.A}{det.A}$

$adj.A = (cof.A)^T$

$cof.A = (-1)^{i+j}$(mainer)

Ex: solve the following set of linear equations by matrix invers method:

$2x_1 + x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 6$

$x_1 + 2x_2 + x_3 = 3$

sol.:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad , b = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

$X = b * A^{-1}$

$A^{-1} = \frac{adj.A}{det.A}$

$$m = \begin{matrix} m11 & m12 & m13 \\ m21 & m22 & m23 \\ m31 & m32 & m33 \end{matrix}$$

m11 = $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$ = (-1-2) = -3

m12 = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ = (1-1) = 0

m13 = $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ = (2+1) = 3

m21 = $\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ = (1+2) = 3

m22 = $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ = (2+1) = 3

m23 = $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ = (4-1) = 3

m31 = $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ = (1-1) = 0

m32 = $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ = (2+1) = 3

m33 = $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ = (-2-1) = -3

$$m = \begin{bmatrix} -3 & 0 & 3 \\ 3 & 3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$cof.A = (-1)^{i+j}.m = \begin{bmatrix} -3 & 0 & 3 \\ -3 & 3 & -3 \\ 0 & -3 & -3 \end{bmatrix}$$

$$adj.A = cof.A^T = \begin{bmatrix} -3 & -3 & 0 \\ 0 & 3 & -3 \\ 3 & -3 & -3 \end{bmatrix}$$

Det.A = $2\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} - 1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ = -9

$$A^{-1} = \frac{adj.A}{det.A}$$

$$A^{-1} = \frac{\begin{bmatrix} -3 & -3 & 0 \\ 0 & 3 & -3 \\ 3 & -3 & -3 \end{bmatrix}}{-9}$$

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 0 & -1/3 & 1/3 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}$$

$x = A^{-1} * b$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 0 & -1/3 & 1/3 \\ -1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1(0)}{3} + & \frac{1}{3}(6) + & 0(3) \\ 0(0) + & -\frac{1}{3}(6) + & \frac{1}{3} \quad (3) \\ -\frac{1}{3}(0) + & \frac{1}{3}(6) + & \frac{1}{3}(3) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

X1 =2

X2 =-1

X3 =3

2. Grammar's Rule:

This method can be used to find the solution for a set of linear equations suppose we have three linear equations:

$a_{11}x1 + a_{12}x2 + a_{13}\ x3 = b1$

$a_{21}x1 + a_{22}x2 + a_{23}x3 = b2$

$a_{31}x1 + a_{32}x2 + a_{33}x3 = b3$

$$A = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}, \quad b = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$

$$X1 = \frac{\begin{bmatrix} b1 & a12 & a13 \\ b2 & a22 & a23 \\ b3 & a32 & a33 \end{bmatrix}}{\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}} = \frac{Det.1}{Det.A}$$

$$X2 = \frac{\begin{bmatrix} a11 & b1 & a13 \\ a21 & b2 & a23 \\ a23 & b3 & a33 \end{bmatrix}}{\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}} = \frac{Det.2}{Det.A}$$

$$X3 = \frac{\begin{bmatrix} a11 & a12 & b1 \\ a21 & a22 & b2 \\ a31 & a32 & b3 \end{bmatrix}}{\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}} = \frac{Det.3}{Det.A}$$

Ex: solve the following equations using Grammar's rule :.

$\mathbf{2x_1 + 4x_2 + 2x_3 = 16}$

$\mathbf{2x_1 - x_2 - 2x_3 = -6}$

$\mathbf{4x_1 + x_2 - 2x_3 = 0}$

$$\begin{bmatrix} 2 & 4 & 2 \\ 2 & -1 & -2 \\ 4 & 1 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 16 \\ -6 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 \\ 2 & -1 & -2 \\ 4 & 1 & -2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 16 \\ -6 \\ 0 \end{bmatrix}$$

$$Det.A = 2\begin{pmatrix} -1 & -2 \\ 1 & -2 \end{pmatrix} - 4\begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix} + 2\begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}$$

Det.A = 2(2+2) -4(-4+8) +2(2+4) = 8 – 16 + 12 = 4

$$A1 = \begin{bmatrix} \mathbf{16} & 4 & 2 \\ \mathbf{-6} & -1 & -2 \\ \mathbf{0} & 1 & -2 \end{bmatrix}, \quad A2 = \begin{bmatrix} 2 & \mathbf{16} & 2 \\ 2 & \mathbf{-6} & -2 \\ 4 & \mathbf{0} & -2 \end{bmatrix}, \quad A3 = \begin{bmatrix} 2 & 4 & \mathbf{16} \\ 2 & -1 & \mathbf{-6} \\ 4 & 1 & \mathbf{0} \end{bmatrix}$$

$$Det.1 = 16\begin{pmatrix} -1 & -2 \\ 1 & -2 \end{pmatrix} - 4\begin{pmatrix} -6 & -2 \\ 0 & -2 \end{pmatrix} + 2\begin{pmatrix} -6 & -1 \\ 0 & 1 \end{pmatrix} = 64 – 48 – 12 = 4$$

$$Det.2 = 2\begin{pmatrix} -6 & -2 \\ 0 & -2 \end{pmatrix} - 16\begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix} + 2\begin{pmatrix} 2 & -6 \\ 4 & 0 \end{pmatrix} = 24 - 64 + 48 = 8$$

$$\text{Det.3} = 2\begin{pmatrix} -1 & -6 \\ 1 & 0 \end{pmatrix} - 4\begin{pmatrix} 2 & -6 \\ 4 & 0 \end{pmatrix} + 16\begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix} = 12 - 96 + 96 = 12$$

$$x_1 = \frac{Det.1}{Det.A} = \frac{4}{4} = 1$$

$$x_2 = \frac{Det.2}{Det.A} = \frac{8}{4} = 2$$

$$x_3 = \frac{Det.3}{Det.A} = \frac{12}{4} = 3$$

3. Jacobi method:

In this method we will arrange the equations form with these conditions:

$|a_{11}| > |a_{12}|$ and $|a_{13}|$

$|a_{22}| > |a_{21}|$ and $|a_{23}|$

$|a_{33}| > |a_{31}|$ and $|a_{32}|$

$$\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$$

Then rearrange the equations with form

x1= f(x2,x3) ……………….(equ.1)

x2 =f(x1,x3) ……………….(equ.2)

x3 =f(x1,x2) ……………….(equ.3)

To find x1,x2,x3in this first iteration we input x1,x2,x3in the right hand equal =0

To find x1,x2,x3in the second iteration we input x1,x2,x3(which is found in first iteration) in the right hand still until reach error or same variable values .

Example: solve the following set of linear algebraic equations using Jacobi method.

$x_1 + 4x_2 - 2x_3 = 3$

$5x_1 - 2x_2 + x_3 = 4$

$x_1 + 2x_2 + 4x_3 = 17$

solution:

$5x_1 - 2x_2 + x_3 = 4$

$x_1 + 4x_2 - 2x_3 = 3$

$x_1 + 2x_2 + 4x_3 = 17$

from equation (1) $x_1 = \frac{4}{5} + \frac{2}{5}x2 - \frac{1}{5}x3$

from equation (2) $x_2 = \frac{3}{4} - \frac{1}{4}x1 + \frac{2}{4}x3$

from equation (3) $x_3 = \frac{17}{4} - \frac{1}{4}x1 - \frac{2}{4}x2$

suppose x1,x2,x3 ithe right hand side =0

$x_1 = \frac{4}{5} + \frac{2}{5}(0) - \frac{1}{5}(0)$ **x1 =0.8**

$x_2 = \frac{3}{4} - \frac{1}{4}(0) + \frac{2}{4}(0)$ **x2 = 0.75**

$x_3 = \frac{17}{4} - \frac{1}{4}(0) - \frac{2}{4}(0)$ **x3 = 4.25**

second iteration

$x_1 = \frac{4}{5} + \frac{2}{5}(0.75) - \frac{1}{5}(4.25) =$ **0.25**

$x_2 = \frac{3}{4} - \frac{1}{4}(0.8) + \frac{2}{4}(4.25) =$ **2.69**

$x_3 = \frac{17}{4} - \frac{1}{4}(0.8) - \frac{2}{4}(0.75) =$ **3.68**

third iteration

$x_1 = \frac{4}{5} + \frac{2}{5}(2.69) - \frac{1}{5}(3.68) =$ **1.441**

$x_2 = \frac{3}{4} - \frac{1}{4}(0.25) + \frac{2}{4}(3.68) =$ **2.53**

$x_3 = \frac{17}{4} - \frac{1}{4}(0.25) - \frac{2}{4}(2.69) =$ **2.85**

forth iteration

$x_1 = \frac{4}{5} + \frac{2}{5}(2.53) - \frac{1}{5}(2.85) =$

$x_2 = \frac{3}{4} - \frac{1}{4}(\mathbf{1.441}) + \frac{2}{4}(\mathbf{2.85}) =$

$x_3 = \frac{17}{4} - \frac{1}{4}(\mathbf{1.441}) - \frac{2}{4}(\mathbf{2.53}) =$

still until reach error or same variable values.

4. Gauss-siedel method

In this method the equations are arrange with condition same the previous method **Jacobi** . arrange new equations with form :

x1= f(x2,x3) ……………….(equ.1)

x2 =f(x1,x3) ……………….(equ.2)

x3 =f(x1,x2) ……………….(equ.3)

suppose x1=x2=x3=0 , substitute all variable in equation(1) and remember to substitute any new evaluated variable value in equation that's follow.

Example: solve the following set of linear algebraic equations using Gauss-seidel method.

$x_1 + 10x_2 + 2x_3 = 16$

$2x_1 + 3x_2 - 8x_3 = 1$

$11x_1 + 2x_2 + x_3 = 15$

Solution:

Arrange

$11x_1 + 2x_2 + x_3 = 15$

$x_1 + 10x_2 + 2x_3 = 16$

$2x_1 + 3x_2 - 8x_3 = 1$

from equation (1) $x_1 = \frac{15}{11} - \frac{2}{11}x2 - \frac{1}{11}x3$

from equation (2) $x_2 = \frac{16}{10} - \frac{1}{10}x1 - \frac{2}{10}x3$

from equation (3) $x_3 = \frac{-1}{8} + \frac{2}{8}x1 + \frac{3}{8}x2$

x1 =x2=x3=0

x_1 =1.3636 – 0.1818x2 – 0.0909x3

$x2$ = 1.6 – 0.1x1 -0.2x3

$x3$ = -0.125 + 0.25x1 + 0.375x2

x_1 =1.3636 – 0.1818(0) – 0.0909(0), **x1= 1.3636**

$x2$ = 1.6 – 0.1(1.3636) -0.2(0), **x2 = 1.4636**

$x3$ = -0.125 + 0.25(1.3636) + 0.375(1.4636), **x3 = 0.7648**

x_1 =1.3636 – 0.1818(1.4636) – 0.0909(0.7648), **x1= 1.028**

$x2$ = 1.6 – 0.1(1.082) -0.2(0.7648), **x2 = 1.344**

$x3$ = -0.125 + 0.25(1.028) + 0.375(1.344), **x3 = 0.636**

Nonlinear set of equation

The problem to be solved can be written as:

F1(x1,x2,x3,……..xn) = 0

F2(x1,x2,x3,……..xn) = 0

Fn(x1,x2,x3,…….xn) = 0

Where each function F:(x1,x2,x3,……..xn) corresponds to a nonlinear function containing one or more of the variables whose values are known.

1. Simple iteration or Gauss –seidel (iteration method)

Some steps in this method which are :

Rearrangeing

x1 = F1(x1,x2,x3,……..xn) = 0

x2 = F2(x1,x2,x3,……..xn) = 0

x3 = Fn(x1,x2,x3,…….xn) = 0

select starting values $x_1^0, x_2^0, x_3^0, \ldots\ldots x_n^0$,continues until reach the error or $x_i = x_i^0$

$E = | x_1 - x_1^0|$, $I = 1,2,3,\ldots,n$

Example: solve the following equations:

$$F1(x1, x2) = \frac{1}{2}\sin(x1x2) - \frac{x2}{4\pi} - \frac{x1}{2} = 0$$

$$F2(x1, x2) = (1- \frac{1}{4\pi})(e^{2x1} - e) + \frac{e * x2}{\pi} -2ex1 = 0$$

$x_1^0 = 0.4$, $x_2^0 = 3$

solution:

$$\frac{1}{2}\sin(x1x2) - \frac{x2}{4\pi} = \frac{x1}{2}$$

$$-(1- \frac{1}{4\pi})(e^{2x1} - e) + 2ex1 = \frac{e*x2}{\pi}$$

$$x1 = \sin(x1x2) - \frac{x2}{2\pi}$$

$$x2 = 2\pi x1 - (\pi - \frac{1}{4})(\frac{e^{2x1}}{\pi} -1)$$

starting with $x_1^0 = 0.4$, $x_2^0 = 3$ (radian)

x_1^0	x_2^0	x_1	x_2
0.4	3	0.454	3.036
0.454	3.036	0.499	3.105
0.499	3.105	0.505	3.139
0.505	3.139	0.5	3.142
0.5	3.142	0.5	3.140
0.5	3.140	0.5	3.140

X1 = 0.5, x2 = 3.140

2. Newton –Raphson method

Suppose the following two independent equations in two variables x1, x2

F1(x1, x2) = 0

F2(x1, x2) = 0

To apply Newton-Raphson method expand each equation as a first order Taylor series to get a set of linear equations.

$$0 = F1(x_1, x_2) = F1(x_1^0, x_2^0) + \frac{\partial F1(x10,x20)}{\partial x1}(x_1 - x_1^0) + \frac{\partial F1(x10,x20)}{\partial x2}(x_2 - x_2^0)$$

$$F(x) = F(x0) + h\frac{dF(x0)}{dx}$$

h = interval $(x_i - x_i^0)$

$$0 = F2(x_1, x_2) = F2(x_1^0, x_2^0) + \frac{\partial F2(x10,x20)}{\partial x1}(x_1 - x_1^0) + \frac{\partial F2(x10,x20)}{\partial x2}(x_2 - x_2^0)$$

Suppose ;

1- the partial derivative Fij

2- $xi - xi^0 = \Delta xi$

$$F11\ \Delta x1 + F12\ \Delta x2 = -F1(x_1^0, x_2^0)$$

$$F21\ \Delta x1 + F22\ \Delta x2 = -F2(x_1^0, x_2^0)$$

$$\begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix}\begin{pmatrix} \Delta x1 \\ \Delta x2 \end{pmatrix} = \begin{pmatrix} -F1 \\ -F2 \end{pmatrix}$$

Repeat the above example by Newton-Raphson method.

$$F1(x1, x2) = \frac{1}{2}\sin(x_1.x_2) - \frac{x2}{4\pi} - \frac{x1}{2} = 0$$

$$F2(x1, x2) = (1 - \frac{1}{4\pi})(e^{2x1} - e) + \frac{e * x2}{\pi} - 2ex1 = 0$$

$x_1^0 = 0.4, \qquad x_2^0 = 3$

$(0.3)\ e^{2x1} - (0.3)\ e$

$$F1 = F1(x_1^0, x_2^0) = \frac{1}{2}\sin(0.4*3) - \frac{3}{4\pi} - \frac{0.4}{2} = 0.0273$$

$$F2 = F2(x_1^0, x_2^0) = (1 - \frac{1}{4\pi})(e^{2*0.4} - e) + \frac{e * 3}{\pi} - 2*e*0.4 = -0.0324$$

$$F11 = \frac{\partial F1}{\partial x1}(x_1^0, x_2^0) = \frac{x2\cos(x01.x02)}{2} - \frac{1}{2} = 3/2\cos(0.4*3) - \frac{1}{2} = 0.0435$$

$$F12 = \frac{\partial F1}{\partial x2}(x_1^0, x_2^0) = \frac{x1\cos(x01x02)}{2} - \frac{1}{4\pi} = \frac{0.4\cos(0.4*3)}{2} - \frac{1}{4*\pi} = -0.0071$$

$F21 = \frac{\partial F2}{\partial x1}(x_1^0, x_2^0) = (2 - \frac{1}{2\pi})e^{2x1} - 2e = -1.3$

$F22 = \frac{\partial F2}{\partial x2}(x_1^0, x_2^0) = \frac{e}{\pi} = \frac{2.7183}{3.14} = 0.868$

$$\begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix}\begin{pmatrix} \Delta x1 \\ \Delta x2 \end{pmatrix} = \begin{pmatrix} -F1 \\ -F2 \end{pmatrix}$$

$$\begin{pmatrix} 0.0435 & -0.0071 \\ -1.3 & 0.865 \end{pmatrix}\begin{pmatrix} \Delta x1 \\ \Delta x2 \end{pmatrix} = \begin{pmatrix} -0.0273 \\ 0.0324 \end{pmatrix}$$

$A = \begin{pmatrix} 0.0435 & -0.0071 \\ -1.3 & 0.865 \end{pmatrix}$, $b = \begin{pmatrix} -0.0273 \\ 0.0324 \end{pmatrix}$

Determinant of matrix A =[(0.0435 *0.865) – (-1.3 *(-0.0071))] =

|A| =0.0376 – 0.00923 = 0.02837

$A1 = \begin{pmatrix} -0.0273 & -0.0071 \\ 0.0324 & 0.865 \end{pmatrix}$, $A2 = \begin{pmatrix} 0.0435 & -0.0273 \\ -1.3 & 0.0324 \end{pmatrix}$

DA1= -0.0234

DA2 = -1.301

$\Delta x1 = \frac{DA1}{DA} = -0.83178$

$\Delta x2 = \frac{DA2}{DA} = -1.25$

$(x_1 - x_1^0) = -0.8318$

$(x_1 - 0.4) = -0.8318$

$(x_2 - 3) = -1.25$

X1 = - 0.4318, x2 = 1.75

H.W

Find the solution for for nonlinear equation system below :

$F1(x_1, x_2) = x_1^2 + x_2^2 e^{-5x1} - 2$

$F2(x_1, x_2) = x_2^2 + x_1^2 e^{-7x2} - 1$

$x_1^0 = 3$, $x_2^0 = 2$

Numerical solution of ordinary differential equation

1. Tayler series

This method apply for problem involving having an initial value

The Numerical solution is:

$$y(x) = y(_0) + h\bar{y}(_0) + \frac{h^2}{2!}\bar{\bar{y}}(_0) + \frac{h^3}{3!}\bar{\bar{\bar{y}}}(_0) + \frac{h^4}{4!}\bar{\bar{\bar{\bar{y}}}}(_0) + \ldots\ldots\ldots$$

or:

$$f(x_0 + h) = f(x_0) + h\bar{f}(x_0) + \frac{h^2}{2!}\bar{\bar{f}}(x_0) + \frac{h^3}{3!}\bar{\bar{\bar{f}}}(x_0) + \frac{h^4}{4!}\bar{\bar{\bar{\bar{f}}}}(x_0) + \ldots\ldots\ldots$$

where:

h: is the interval used ($x - x_0$)

x_0: is the initial value of x

$\bar{f}$: first order differential equation

$\bar{\bar{f}}$: second order differential equation

$\bar{\bar{\bar{f}}}$: third order differential equation

$\bar{\bar{\bar{\bar{f}}}}$: forth order differential equation

Example: using Tayler series to solve the following ordinary differential equation

$$\frac{dy}{dx} = x + y$$

(find y(0.3))

Where: [$y(_0) = 0$ ($x = 0$, $y = 0$)] , h =0.1 .

Solution:

x	y
0	0
0.1	?
0.2	?

0.3	?

Find y at x = 0.3 for h = 0.1

$$\frac{dy}{dx} = x + y$$

$$\overline{y} = x + y$$

$$\overline{\overline{y}} = 1 + \overline{y}$$

$$\overline{\overline{\overline{y}}} = \overline{\overline{y}}$$

$$\overline{\overline{\overline{\overline{y}}}} = \overline{\overline{\overline{y}}}$$

$$y(x) = f(x_0 + h) = y(_0) + h\overline{y}(_0) + \frac{h^2}{2!}\overline{\overline{y}}(_0) + + \frac{h^3}{3!}\overline{\overline{\overline{y}}}(_0) + \frac{h^4}{4!}\overline{\overline{\overline{\overline{y}}}}(_0) + \ldots\ldots\ldots$$

$y(_0) = 0$ ($x_0 = 0$, $y_0 = 0$)

$\overline{y}(_0) = x_0 + y_0 = 0 + 0 = 0$

$\overline{\overline{y}} = 1 + \overline{y}(0) = 1 + 0 = 1$

$\overline{\overline{\overline{y}}} = \overline{\overline{y}} = 1$

$\overline{\overline{\overline{\overline{y}}}} = \overline{\overline{\overline{y}}} = 1$

$$y(0.1) = f(0 + 0.1) = 0 + 0.1(0) + \frac{0.1^2}{2*1}(1) + \frac{0.1^3}{3*2*1}(1) + \frac{0.1^4}{4*3*2}(1)$$

$y(0.1) = 0.005$

$x_0 = 0.1,\quad y_0 = 0.005$

x	y
0	0
0.1	0.005
0.2	?

$\overline{y} = x + y = 0.1 + 0.005 = 0.105$

$\overline{\overline{y}} = 1 + \overline{y} = 1 + 0.105 = 1.105$

$\overline{\overline{\overline{y}}} = \overline{\overline{y}} = 1.105$

$\overline{\overline{\overline{\overline{y}}}} = \overline{\overline{\overline{y}}} = 1.105$

$y(x) = f(x_0 + h) = y(0.2) = f(0.1 + 0.1)$

$y(0.2) = y(_0) + h\bar{y}(_0) + \frac{h^2}{2!}\bar{\bar{y}}(_0) + + \frac{h^3}{3!}\bar{\bar{\bar{y}}}(_0) + \frac{h^4}{4!}\bar{\bar{\bar{y}}}(_0)$

$y(0.2) = 0.005 + 0.1(0.105) + \frac{0.1^2}{2*1}(1.105) + \frac{0.1^3}{3*2*1}(1.105) + \frac{0.1^4}{4*3*2*}(1.105)$

$y(0.2) = 0.021$

$x_0 = 0.2, \quad y_0 = 0.021$

$\bar{y} = x + y = 0.2 + 0.021 = 0.221$

$\bar{\bar{y}} = 1 + \bar{y} = 1 + 0.221 = 1.221$

$\bar{\bar{\bar{y}}} = \bar{\bar{y}} = 1.221$

$\bar{\bar{\bar{\bar{y}}}} = \bar{\bar{\bar{y}}} = 1.221$

$y(0.3) = 0.021 + 0.1(0.221) + \frac{0.1^2}{2*1}(1.221) + \frac{0.1^3}{3*2*1}(1.221) + \frac{0.1^4}{4*3*2*}(1.221)$

$y(0.3) = 0.049$

Example: $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, $\quad h = 0.2$ find $y(0.6)$

x	y
0	1
0.2	?
0.4	?
0.6	?

$y = ?$, at $x = 0.6$

$\bar{y} = x^2 + y^2 = 0^2 + 1^2 = 1$

$\bar{\bar{y}} = 2x + 2y\bar{y} = 2*0 + 2*1*1 = 2$

$\bar{\bar{\bar{y}}} = 2 + 2[y\bar{\bar{y}} + \bar{y}\,\bar{y}] = 2 + 2[1*2 + 1*1] = 8$

$\bar{\bar{\bar{y}}} = 2 + 2y\bar{\bar{y}} + 2(\bar{y})^2 = 2 + 2*1*2 + 2*1^2 = 8$

$\bar{\bar{\bar{\bar{y}}}} = 2[y\bar{\bar{\bar{y}}} + \bar{\bar{y}}\,\bar{y}] + 4\bar{y}\bar{\bar{y}} = 2[1*8 + 2*1] + 4*1*2 = 28$

$y(x) = y(_0) + h\bar{y}(_0) + \frac{h^2}{2!}\bar{\bar{y}}(_0) + \frac{h^3}{3!}\bar{\bar{\bar{y}}}(_0) + \frac{h^4}{4!}\bar{\bar{\bar{y}}}(_0)$

$y(0.2) = 1 + 0.2(1) + \frac{0.2^2}{2*1}(2) + \frac{0.2^3}{3*2*1}(8) + \frac{0.2^4}{4*3*2*}(28)$

$y(0.2) = 1.252$

$x_0 = 0.2, \quad y_0 = 1.252$

$\bar{y} = x^2 + y^2 = 0.2^2 + 1.252^2 = 1.6$

$\bar{\bar{y}} = 2x + 2y\bar{y} = 2*0.2 + 2*1.252*1.6 = 4.4$

$\bar{\bar{\bar{y}}} = 2 + 2[y\bar{\bar{y}} + \bar{y}\,\bar{y}] = 2 + 2[1.252*4.4 + 1.6*1.6] = 18.137$

$\bar{\bar{\bar{y}}} = 2 + 2y\bar{\bar{y}} + 2\bar{y}^2 = 2 + 2*1.252*4.4 + 2*1.6^2 =$

$\bar{\bar{\bar{\bar{y}}}} = 2[y\bar{\bar{\bar{y}}} + \bar{\bar{y}}\,\bar{y}] + 4\bar{y}\bar{\bar{y}} = 2[1.252*18.137 + 4.4*1.6] + 4*1.6*4.4 = 87.65$

$y(0.4) = 1.252 + 0.2(1.6) + \frac{0.2^2}{2*1}(4.4) + \frac{0.2^3}{3*2*1}(18.137) + \frac{0.2^4}{4*3*2*}(87.65)$

$y(0.4) = 1.252 + 0.12 + 0.088 + 0.024 + 0.0058$

$y(0.4) = 1.4898$

$x_0 = 0.4, \quad y_0 = 1.4898$

$\bar{y} = x^2 + y^2 = 0.4^2 + 1.4898^2 = 2.379$

$\bar{\bar{y}} = 2x + 2y\bar{y} = 2*0.4 + 2*1.4898*2.379 = 14.910$

$\bar{\bar{\bar{y}}} = 2 + 2[y\bar{\bar{y}} + \bar{y}\,\bar{y}] = 2 + 2[1.4898*14.910 + 2.379*2.379] = 57.745$

$\bar{\bar{\bar{\bar{y}}}} = 2[y\bar{\bar{\bar{y}}} + \bar{\bar{y}}\,\bar{y}] + 4\bar{y}\bar{\bar{y}} = 2[1.4898*57.745 + 14.910*2.379] + 4*2.379*14.910 = 384.88$

$y(0.6) = 1.4898 + 0.2(2.379) + \frac{0.2^2}{2*1}(14.910) + \frac{0.2^3}{3*2*1}(57.745) + \frac{0.2^4}{4*3*2*}(384.88)$

$y(0.6) = 2.366$

2. Euler's method

in this method we can find the solution of ordinary differential equation by using:

$y(x) = y(_0) + h\bar{y}(_0)$

$f(x_0 + h) = f(x_0) + h\bar{f}(x_0)$

Example: using Euler's method to solve the following ordinary differential equation

$$\frac{dy}{dx} = x + y$$

(find y(0.3))

Where: [$y_{(0)} = 0$ (x = 0, y = 0)] , h =0.1 .

Solution:

x	y
0	0
0.1	?
0.2	?
0.3	?

Find y at x = 0.3 for h = 0.1

$\bar{y} = x + y = 0+0$

$y(x) = y_{(0)} + h\bar{y}_{(0)}$

$y(0.1) = 0 + 0.1 * 0 = 0$

$y(0.1) = 0$

$x_0 = 0.1 \quad y_0 = 0$

$y(x) = y_{(0)} + h\bar{y}_{(0)}$

$\bar{y} = x + y = 0.1 + 0 = 0.1$

$y(x) = y_{(0)} + h\bar{y}_{(0)}$

$y(0.2) = 0 + 0.1*0.1 = 0.01$

$x_0 = 0.2 \quad y_0 = 0.01$

$\bar{y} = x + y = 0.2 + 0.01 = 0.21$

$y(0.3) = 0.01 + 0.1*0.21 = 0.031$

Example: $\frac{dy}{dx} = (1 + 2x)\sqrt{y}$, **h = 0.25 y(0) = 1 find y =? at x = 1**

Solution:

$y(x_{i+1}) = y(x_i) + h\bar{y}(x_i)$

$y(x) = y(0) + h\bar{y}(0)$

$\bar{y} = (1 + 2x)\sqrt{y}$

$x_0 = 0 \quad y_0 = 1$

$\bar{y}(0) = (1 + 2(0))\sqrt{1} = 1$

$y(0.25) = 1 + 0.25*1 = 1.25$

$x_0 = 0.25 \quad y_0 = 1.25$

$y(x_{i+1}) = y(x_i) + h\bar{y}(x_i)$

$y(0.5) = y(0.25) + h\bar{y}(0.25)$

$\bar{y} = (1 + 2x)\sqrt{y}$

$\bar{y}(0.25) = (1 + 2*0.25)\sqrt{1.25} = 1.677$

$y(0.5) = 1.25 + 0.25*1.677 = 1.669$

$x_0 = 0.5 \quad y_0 = 1.669$

$y(0.75) = y(0.5) + h\bar{y}(0.5)$

$\bar{y}(0.5) = (1 + 2*0.5)\sqrt{1.669} = 2.584$

$y(0.75) = 1.669 + 0.25*2.584 = 2.315$

$x_0 = 0.75 \quad y_0 = 2.315$

$y(1) = y(0.75) + h\bar{y}(0.75)$

$\bar{y}(0.75) = (1 + 2*0.75)\sqrt{2.315} = 3.8$

$y(1) = 1.25 + 0.25*3.8 = 2.2$

3. Modified Euler's method:

In this method there are two steps to find the solution of ordinary differential equation (predictor and corrector):

Step 1

$y(x) = y(_0) + h\overline{y}(_0)$ (predictor)

<u>Step 2</u>

$y(x) = y(0) + \frac{h}{2}[\overline{y}(0) + \overline{y}(x)]$ (corrector)

Example1: using Modified Euler's method to solve the following ordinary differential equation

$$\frac{dy}{dx} = x + y$$

(find y(0.3))

Where: $[y(_0) = 0\ (x_0 = 0,\ y_0 = 0)]$, h =0.1

<u>Solution:</u>

<u>1st iteration</u>

$x_0 = 0,\ y_0 = 0$

$y(x) = y(_0) + h\overline{y}(_0)$ (predictor)

$y(0.1) = 0 + 0.1(0 + 0) = 0$

$x_0 = 0.1,\ y_0 = 0$

$y(x) = y(0) + \frac{h}{2}[\overline{y}(0) + \overline{y}(x)]$ (corrector)

$y(0.1) = y(0) + \frac{h}{2}[\overline{y}(0) + \overline{y}(0.1)]$

$\overline{y}(0) = x + y = 0 + 0 = 0$

$\overline{y}(0.1) = x + y = 0.1 + 0 = 0.1$

$y(0.1) = 0 + \frac{0.1}{2}[0 + 0.1]$

$y(0.1) = 0.005$

<u>2nd iteration</u>

$x_0 = 0.1,\ y_0 = 0.005$

$y(x) = y(_0) + h\overline{y}(_0)$ (predictor)

$y(0.2) = 0.005 + 0.1(0.1 + 0.005) = 0.0155$ ($x_0 = 0.2,\ y_0 = 0.015$)

$y(x) = y(0) + \frac{h}{2}[\overline{y}(0) + \overline{y}(x)]$ (corrector)

$y(0.2) = y(0.1) + \frac{h}{2}[\bar{y}(0.1) + \bar{y}(0.2)]$

$\bar{y}(0.1) = x + y = 0.1 + 0.005 = 0.105$

$\bar{y}(0.2) = x + y = 0.2 + 0.0155 = 0.2155$

$y(0.2) = 0.005 + \frac{0.1}{2}[0.105 + 0.2155] = 0.021$

3rd iteration

$x_0 = 0.2, y_0 = 0.021$

$y(x) = y(_0) + h\bar{y}(_0)$ (predictor)

$y(0.3) = 0.021 + 0.1(0.2 + 0.021) = 0.0431$

$y(x) = y(0) + \frac{h}{2}[\bar{y}(0) + \bar{y}(x)]$ (corrector)

$y(0.3) = y(0.2) + \frac{h}{2}[\bar{y}(0.2) + \bar{y}(0.3)]$

$\bar{y}(0.2) = x + y = 0.2 + 0.021 = 0.221$

$\bar{y}(0.3) = x + y = 0.3 + 0.0431 = 0.3431$

$y(0.3) = 0.021 + \frac{0.1}{2}[0.221 + 0.3431] = 0.049$

x_0	y_0 by Taylor	y_0 by Euler's	y_0 by Euler's Modified
0	0	0	0
0.1	0.005	0	0.005
0.2	0.021	0.01	0.021
0.3	0.049	0.031	0.049

Example2: using Modified Euler's method to solve the following ordinary differential equation

$$\frac{dy}{dx} = e^x + y$$

Where: [$y(_0) = 0$ ($x_0 = 0$, $y_0 = 0$)] , h =0.2 find y(0.6)?

Solution:

1st iteration

$x_0 = 0, y_0 = 0$

$y(x) = y(_0) + h\overline{y}(_0)$ (predictor)

$y(0.2) = 0 + 0.2(e^0 + 0) = 0.2$

$y(x) = y(0) + \frac{h}{2}[\,\overline{y}(0) + \overline{y}(x)\,]$ (corrector)

$y(0.2) = y(0) + \frac{h}{2}[\,\overline{y}(0) + \overline{y}(0.2)\,]$

$y(0.2) = 0 + \frac{0.2}{2}[\,(e^0 + 0) + (e^{0.2} + 0.2)\,] = 0.2421$

$y(0.2) = 0.2421$

<u>2nd iteration</u>

$x_0 = 0.2,\ y_0 = 0.2421$

$y(0.4) = y(_{0.2}) + h\overline{y}(_{0.2})$ (predictor)

$y(0.4) = 0.2421 + 0.2(e^{0.2} + 0.2421) = 0.5348$

(corrector)

$y(0.4) = y(0.2) + \frac{h}{2}[\,\overline{y}(0.2) + \overline{y}(0.4)\,]$

$y(0.4) = 0.2421 + \frac{0.2}{2}[\,(e^{0.2} + 0.2421) + (e^{0.4} + 0.5348)\,] = 0.5911$

<u>3rd iteration</u>

$x_0 = 0.4,\ y_0 = 0.5911$

$y(0.6) = y(_{0.4}) + h\overline{y}(_{0.4})$ (predictor)

$y(0.6) = 0.5911 + 0.2(e^{0.4} + 0.5911) = \text{value}$

(corrector)

$y(0.6) = y(0.4) + \frac{h}{2}[\,\overline{y}(0.4) + \overline{y}(0.6)\,]$

$y(0.6) = 0.5911 + \frac{0.2}{2}[\,(e^{0.4} + 0.5911) + (e^{0.6} + \text{value})\,] =$

<u>H.W</u>

1. using Modified Euler's method to solve the following ordinary differential equation

$\frac{dy}{dx}$ = 2x(y – 1)

Where: [y(0) = 0 (x_0 = 0, y_0 = 0)] , h =0.1 find y(0.5)?

2. using Euler's method to solve the following ordinary differential equation

$\bar{y}$ – 2y =3e^x

Where: [y(0) = 0 (x_0 = 0, y_0 = 0)] , h =0.1 find y(0.3)?

4. Runge Kutta method

this method is use to find the solution of ordinary differential equation by:

A. Second order Runge Kutta method:

$\frac{dy}{dx}$ = *f* (x, y)

K1 = h* *f* (x_i, y_i)

K2 = h* *f* ((x_i + h), (y_i + k1))

y(x) = y(0) + $\frac{1}{2}$ (k1 +k2)

Example1: using Second order Runge Kutta method to solve the following ordinary differential equation

$\frac{dy}{dx} = x + y$

(find y(1))

Where: [y(0) = 0 (x_0 = 0, y_0 = 0)] , h =0.2

Solution:

1st iteration

x_0 = 0, y_0 = 0

K1 = h* *f* (x_i, y_i)

K2 = h* *f* ((x$_i$ + h), (y$_i$ + k1))

Y(x) = y(0) + $\frac{1}{2}$ (k1 +k2)

K1 =0.2*(0+0) = 0

K2 =0.2*[(0+0.2) + (0+0)] = 0.04

y(x) = y(0) + $\frac{1}{2}$ (k1 +k2)

y(0.2) = y(0) + $\frac{1}{2}$ (k1 +k2)

y(0.2) = 0 + $\frac{1}{2}$ (0 +0.04)

y(0.2) = 0.02

<u>2nd iteration</u>

x_0 = 0.2, y_0 = 0.02

K1 =0.2*(0.2+0.02) = 0.044

K2 =0.2*[(0.2+0.2) + (0.02+0.044)] = 0.092

y(0.4) = y(0.2) + $\frac{1}{2}$ (k1 +k2)

y(0.4) = 0.02 + $\frac{1}{2}$ (0.044 +0.092)

y(0.4) = 0.0884

X	K1	K2	y
0			0
0.2	0	0.04	0.02
0.4	0.044	0.0928	0.0884
0.6	0.09768	0.157216	0.215848
0.8	0.16317	0.235804	0.415335
1	0.243067	0.33168	0.702708

Second order Runge Kutta method by Excel

	A	B	C	D
1	X	K1	K2	y
2	0			0
3	=A2+0.2	=0.2*(A2+D2)	=0.2*((A2+0.2)+(D2+B3))	=D2+0.5*(B3+C3)

4	نسحب	نسحب	نسحب	نسحب
5				

Example2: using Second order Runge Kutta method to solve the following ordinary differential equation

$\frac{dy}{dx} - y = e^x$

(find y(0.6))

Where: $[y(_0) = 0\ (x_0 = 0,\ y_0 = 0)]$, h =0.2

Solution:

$\frac{dy}{dx} = e^x + y$

1^{st} iteration

$x_0 = 0,\ y_0 = 0$

$K1 = h * f(x_i, y_i)$

$K2 = h * f((x_i + h), (y_i + k1))$

$Y(x) = y(0) + \frac{1}{2}(k1 + k2)$

$K1 = 0.2 * (e^0 + 0) = 0.2$

$K2 = 0.2 * ((e^{0+0.2}) + (0 + 0.2)) = 0.148$

$Y(0.2) = y(0) + \frac{1}{2}(k1 + k2)$

$Y(0.2) = 0 + \frac{1}{2}(0.2 + 0.148) = 0.174$

$x_0 = 0.2,\ y_0 = 0.174$

$K1 = 0.2 * (e^{0.2} + 0.174) = 0.27$

$K2 = 0.2 * ((e^{0.2+0.2}) + (0.174 + 0.27)) = 0.74$

$Y(0.4) = 0.174 + \frac{1}{2}(0.27 + 0.74) = 0.679$

$x_0 = 0.4,\ y_0 = 0.679$

$K1 = 0.2 * (e^{0.4} + 0.679) = 0.43$

$K2 = 0.2 * ((e^{0.4+0.2}) + (0.679 + 0.43)) = 1.47$

$Y(0.6) = 0.679 + \frac{1}{2}(0.43 + 1.47) = 1.629$

B. fourth order Runge Kutta method:

$K1 = h * f(x_i, y_i)$ OR $R1 = h * f(x_i, y_i)$

$K2 = h * f((x_i + \frac{h}{2}), (y_i + \frac{K1}{2}))$ $R2 = h * f((x_i + \frac{\Delta x}{2}), (y_i + \frac{\Delta x}{2} * R1))$

$K3 = h * f((x_i + \frac{h}{2}), (y_i + \frac{K2}{2}))$ $R3 = h * f((x_i + \frac{\Delta x}{2}), (y_i + \frac{\Delta x}{2} * R1))$

$K4 = h * f((x_i + h), (y_i + k3))$ $R4 = h * f((x_i + \Delta x), (y_i + \Delta x . R3))$

$$y(x) = y(0) + \frac{1}{6}(k1 + 2k2 + 2k3 + k4)$$

$$y(x) = y(0) + (\frac{R1}{6} + \frac{R2}{6} + \frac{R3}{6} + \frac{R4}{6} * \Delta x)$$

Example1: using fourth order Runge Kutta method (find y(1.2))

For the following ordinary differential equation

$$\frac{dy}{dx} = x^2 + 2y$$

Where: $[y(_0) = 0 \ (x_0 = 0, y_0 = 0)]$, $h = 0.4$

Solution:

1st iteration

$x_0 = 0, y_0 = 0$

$K1 = h * f(x_i, y_i)$

$K2 = h * f((x_i + \frac{h}{2}), (y_i + \frac{K1}{2}))$

$K3 = h * f((x_i + \frac{h}{2}), (y_i + \frac{K2}{2}))$

$K4 = h * f((x_i + h), (y_i + k3))$

$K1 = 0.4 * (0 \wedge 2 + 2 * 0) = 0$

$K2 = 0.4 * ((0 + \frac{0.4}{2}) \wedge 2 + 2(0 + \frac{0}{2})) = 0.016$

$K3 = 0.4 * ((0 + \frac{0.4}{2}) \wedge 2 + 2(0 + \frac{0.016}{2})) = 0.0224$

$K4 = 0.4 * ((0 + 0.4) \wedge 2 + 2(0 + 0.0224)) = 0.0819$

$y(x) = y(0) + \frac{1}{6}(k1 + 2k2 + 2k3 + k4)$

$y(0.4) = 0 + \frac{1}{6}(0 + 2*0.016 + 2*0.0224 + 0.0819) = 0.02645$

<u>2nd iteration</u>

$x_0 = 0.4$, $y_0 = 0.02645$

$K1 = h* f(x_i, y_i)$

$K2 = h* f((x_i + \frac{h}{2}), (y_i + \frac{K1}{2}))$

$K3 = h* f((x_i + \frac{h}{2}), (y_i + \frac{K2}{2}))$

$K4 = h* f((x_i + h), (y_i + k3))$

K1 = 0.4* (0.4^2 +2*0.02645) = 0.0851

$K2 = 0.4* ((0.4+ \frac{0.4}{2})^2 + 2(0.02645 + \frac{0.0851}{2})) = 0.1992$

$K3 = 0.4* ((0.4+ \frac{0.4}{2})^2 + 2(0.02645 + \frac{0..1992}{2})) = 0.\ 2448$

K4 = 0.4* ((0.4 + 0.4)^2 +2(0.02645 + 0. 2448)) = 0.473

$y(x) = y(0) + \frac{1}{6}(k1 + 2k2 + 2k3 + k4)$

$y(0.8) = 0.02645 + \frac{1}{6}(0.0851 + 2*0.1992 + 2*0.2448 + 0.473) = 0.268$

$x_0 = 0.8$, $y_0 = 0.268$

$K1 = h* f(x_i, y_i)$

$K2 = h* f((x_i + \frac{h}{2}), (y_i + \frac{K1}{2}))$

$K3 = h* f((x_i + \frac{h}{2}), (y_i + \frac{K2}{2}))$

$K4 = h* f((x_i + h), (y_i + k3))$

K1 = 0.4* (0.8^2 +2*0.268) = 0.47

$K2 = 0.4* ((0.8+ \frac{0.4}{2})^2 + 2(0.268 + \frac{1.176}{2})) = 0.8020$

$K3 = 0.4* ((0.8+ \frac{0.4}{2})^2 + 2(0.268 + \frac{1.735}{2})) = 0.9348$

K4 = 0.4* ((0.8 + 0.4)^2 +2(0.268 + 3.271)) = 1.537

$y(1.2) = 0.268 + \frac{1}{6}(1.176 + 2*1.735 + 2*3.271 + 8.518) = 1.181$

X	Y	K1	K2	K3	K4
0.4	0	0.016	0.0224	0.08192	0.026453
0.8	0.085163	0.199228	0.244854	0.473046	0.267515
1.2	0.470012	0.802017	0.934819	1.537867	1.181107

	A	B	C
1	X	K1	K2
2	0		
3	=A2+0.4	=0.4*(A2^2+2*F2)	=0.4*((A2+0.4/2)^2+2*(F2+B3/2))
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	D	E	F
1	K3	K4	y
2	0		0
3	=0.4*((A2+0.4/2)^2+2*(F2+C3/2))	=0.4*((A2+0.4)^2+2*(F2+D3))	=F2+1/6*(B3+2*C3+2*D3+E3)
4	نسحب	نسحب	نسحب

Example:

$\frac{dy1}{dx} = (1 - y_2)\, y_1$

$\frac{dy2}{dx} = (y_1 - 1)\, y_2$

y1(0) =1.5

y2(0) =0.75

h=0.2

find $y_1(1)$, $y_2(1)$ by second order runge-Kutta method

SOLUTION:

$K1 = h * f(x_i, y_i)$

$K2 = h * f((x_i + h), (y_i + k1))$

$y(x) = y(0) + \frac{1}{2}(k1 + k2)$

$K1y1 = h^* f1 (x_i, y_1i + y_2i)$

$K1y2 = h^* f2 (x_i, y_1i + y_2i)$

$K2y1 = h^* f1 ((x_i + h), (y_1i + k_1y_1) + (y_2i + k_1y_2))$

$K2y2 = h^* f2 ((x_i + h), (y_1i + k_1y_1) + (y_2i + k_1y_2))$

$y1(x) = y1(0) + \frac{1}{2}(k_1y_1 + k_2y_1)$

$y2(x) = y2(0) + \frac{1}{2}(k_1y_2 + k_2y_2)$

$K1y1 = \underline{0.2}\ (1 - 0.75)(1.45) = 0.075$

$K1y2 = 0.2(1.5 - 1)(0.75) = 0.075$

$K2y1 = 0.2[1 - (0.75 + 0.075)](1.5 + 0.075) = 0.0551$

$K2y2 = 0.2[\ (1.5 + 0.075) - 1](0.75 + 0.075) = 0.0949$

$y1(0.2) = y1(0) + \frac{1}{2}(k_1y_1 + k_2y_1)$

$y1(0.2) = 1.5 + \frac{1}{2}(0.075+0.0551) = \underline{1.45651}$

$y2(x) = y2(0) + \frac{1}{2}(k_1y_2 + k_2y_2)$ **at x = 0.2**

$y2(0.2) = 0.75 + \frac{1}{2}(0.075 + 0.0949) = \underline{0.835}$

y1 =1.45651, y2 = 0.835 x = 0.2

continue to find **y1(0.4)** and **y2(0.4)**

second and higher order (ODE) by using Euler's method

$\bar{\bar{y}} = f(x, y, \bar{y})$

$\bar{\bar{\bar{y}}} = f(x, y, \bar{y}, \bar{\bar{y}})$

Or I general

$\frac{d^n y}{dx^n} = f(x, y, \bar{y}, \bar{\bar{y}}, \dots y^{n-1})$

Suppose $P = \bar{y}$, $\frac{dP}{dx} = \bar{P} = \bar{\bar{y}}$

$\bar{\bar{y}} = f(x, y, \bar{y})$

$\bar{P} = f(x, y, P)$

Ex: by using Euler method find y(1) for the following

x	y	$P = \bar{y}$
0	0	0
0.2		
0.4		
0.6		
0.8		
1		

$\bar{\bar{y}} = \frac{1}{2}(x + y + \bar{y} + 2)$

Where: $y(0) = 0$, $\bar{y}(0) = 0$, $\Delta x = 0.2 = h$

SOLUTION:

Suppose $P = \bar{y} \leftrightarrow \bar{P} = \bar{\bar{y}}$

$\bar{P} = \frac{1}{2}(x + y + p + 2)$

$y(x) = y(_0) + h\bar{y}(_0)$

$y(x) = y(_0) + hp(_0)$

$p(x) = p(_0) + h\bar{p}(_0)$

$y(_0) = 0$, $\bar{y}(_0) = 0 = P(0)$, $\Delta x = 0.2 = h$

$y(0.2) = y(_0) + h\bar{y}(_0)$

$y(0.2) = 0 + 0.2*(0) = 0$

X	Y	$P = \bar{y}$
0	0	0
0.2	0	0.2
0.4		
0.6		
0.8		
1		

$p(x) = p(_0) + h\bar{p}(_0)$

$p(0.2) = 0 + 0.2\left[\frac{1}{2}(0 + 0 + 0 + 2)\right]$

$p(0.2) = 0.2$

at $x = 0.2$, $y = 0$, $P = 0.2$

$y(0.4) = y(_{0.2}) + h\,P(_{0.2})$

$y(0.4) = 0 + 0.2 * 0.2 = 0.04$

$p(0.4) = p(_{0.2}) + h\bar{p}(_{0.2})$

$p(0.4)$ $= 0.2 + 0.2\left[\frac{1}{2}(0.2 + 0 + 0.2 + 2)\right]$

$p(0.4)$ $= 0.44$

X	Y	$P = \bar{y}$
0	0	0
0.2	0	0.2
0.4	0.04	0.44
0.6		
0.8		
1		

at $x = 0.4$, $y = 0.04$, $P = 0.44$

y(0.6) = y($_{0.4}$) + h P($_{0.4}$)

y(0.6) = 0.04 + 0.2 * 0.44 = 0.13

p(0.6) = p($_{0.4}$) + h$\bar{p}$($_{0.4}$)

p(0.4) =0.44 + 0.2[$\frac{1}{2}$ (0.4 + 0.04 + 0.44+2)]

p(0.4) = 0.728

X	Y	P =$\bar{y}$
0	0	0
0.2	0	0.2
0.4	0.04	0.44
0.6	0.13	0.728
0.8	0.27	1.074
1	0.49	1.48

Ex: given the van der pol equation is a model of an electronic circuit that arose back in the days of vacuum tubes. by using Euler method find

$\bar{y}$ (0.4) P(0.4) for the following:

$$\frac{d^2y}{dx^2} - (1 - y^2)\frac{dy}{dx} + y = 0 \quad \leftrightarrow \bar{p} = \bar{\bar{y}} = (1 - y^2)\,\bar{y} - y$$

Where: y(0) = 0, $\bar{y}$(0) = 1, h= 0.2

Solution: p =$\bar{y}$ ↔$\bar{p}$ =$\bar{\bar{y}}$

y(x) = y($_0$) + h$\bar{y}$($_0$)

y(0.2) = 0 + 0.2(1) =0.2

p(x) = p($_0$) + h$\bar{p}$($_0$)

p(0.2) = 1 + 0.2[**(1 - 0^2)**1 – 0) = 1.2

at x = 0.2, y = 0.2, P=$\bar{y}$=1.2

y(0.4) = y($_{0.2}$) + h$\bar{y}$($_{0.2}$)

y(0.4) = 0.2 + 0.2(1.2) =0.44

p(0.4) = p($_{0.2}$) + h$\bar{p}$($_{0.2}$)

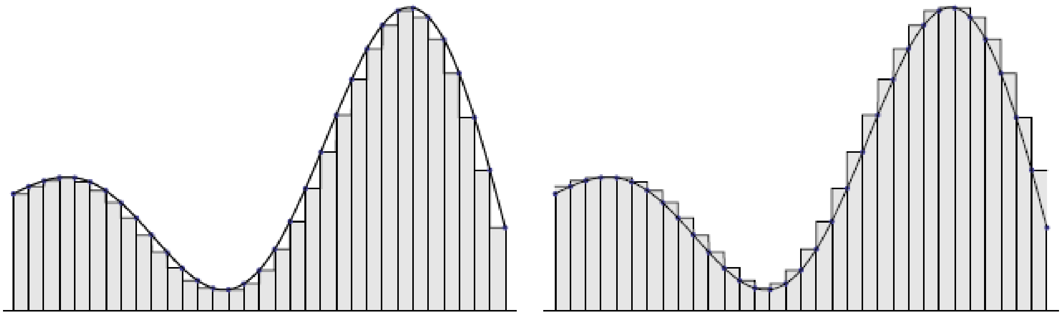
$\bar{y}$ (0.4) = $\bar{y}$ ($_{0.2}$) + h$\bar{\bar{y}}$($_{0.2}$)

p(0.4) = 1.2 + 0.2[**(1 - 0.2^2)**1.2 – 0.2) = 1.39

Numerical integration

1- Rectangular rule method

Area = $\int_{x1}^{xn} y \cdot dx$



= $\sum_{i=1}^{n} yi$**(x**$_{i+1}$ **– x**$_i$**), h=(x**$_{i+1}$ **– x**$_i$**)**

$\sum_{i=1}^{n} yi \cdot h$ **= h**$\sum_{i=1}^{n} yi$ **= h(y1+y2+y3+…..yn)**

$$h = \frac{xn - x1}{N}$$

Example: find area under the curve between x=50 and x=200 for

$y = \mathbf{0.}04e^{\frac{-(x-90)^2}{200}} + 0.2e^{\frac{-(x-130)^2}{800}}$

$\int_{50}^{200} .\mathbf{0.}04e^{\frac{-(x-90)^2}{200}} + 0.2e^{\frac{-(x-130)^2}{800}}$

h= 10 = Δx

solution:

x	50	60	70	80	90	100	...	…	200
y	$2.012*10^{-5}$	0.0048	0.005655						

$\int_{x1}^{xn} y \cdot dx$ = h $\sum y$ = 2.005216

Example: Evaluate

$\int_0^{\frac{\pi}{2}} x^2$.cosx.dx

N =6

h $= \frac{xn-x0}{N} = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12}$

0	0.2617	0.5235	0.785	1.047	1.3089	1.57
y_0	y_1	y_2	y_3	y_4	y_5	y_6
0	0.0662	0.237	0.436	0.548	0.4435	-1.806

Radin

Area = h ($y_0 + y_1 + y_2 + y_3 + y_4 + y_4 + y_6$)

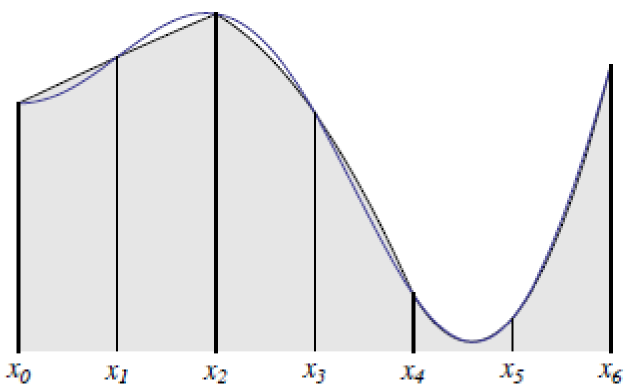
$=\frac{\pi}{12}$ (0+0.0662+0.237+0.436+0.548+0.4435-1.806)

Area=?

2- Trapezoidal rule method

Area = $\int_{x1}^{xn} y \cdot dx$

$= \sum_{i=1}^{n} \frac{y(i+1)+yi}{2} \cdot (x_{i+1} - x_i),$



(b)

$\frac{h}{2}\sum_{i=1}^{n} y(i+1) + yi$

Area = h $[\frac{y0+y1}{2} + \frac{y1+y2}{2} + \frac{y2+y3}{2} + \frac{y3+y4}{2} + \ldots\ldots\ldots + \frac{y(n-1)+yn}{2}]$

Area = $\frac{h}{2}$ [$y_0 + y_n + 2\sum_{i=2}^{n-1} yi$]

Example: Evaluate

$\int_0^{\frac{\pi}{2}} x^2$.cosx.dx

N =6

h $=\frac{xn-x0}{N} = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
0	0.2617	0.5235	0.785	1.047	1.3089	1.57

y_0	y_1	y_2	y_3	y_4	y_5	y_6
0	0.0662	0.237	0.436	0.548	0.4435	-1.806

Area = $\frac{h}{2}$ [y_0 +y_6 +2(y1 + y_2 + y_3 + y_4 + y_5)]

Area = $\frac{\frac{\pi}{12}}{2}$ [0 – 1.806 +2(0.0662 + 0.327 + 0.436 + 0.548 + 0.4435)]

Area = 0.4533

Ex: by trapezoidal rule evaluate the integral

$\int_0^{\pi} \sin(x)$.dx,   N=6

Sol: $\int_0^{\pi} \sin(x)$.dx =$[-cosx]_0^{\pi}$ = 2

$\int_0^{\pi} \sin(x)$.dx =$\frac{\boldsymbol{h}}{\boldsymbol{2}}$[$y_0$ +y_6 +2(y_1 + y_2 + y_3 + y_4 + y_5)]

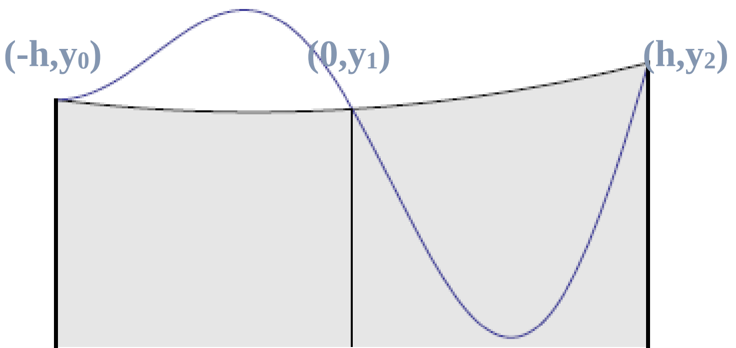
h =$\frac{xn-x0}{N}$ = $\frac{\pi - 0}{6}$ = $\frac{\pi}{6}$ y =sinx

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
y_0	y_1	y_2	y_3	y_4	y_5	y_6
0	0.5	0.866	1	0.5	0.5	0

=$\frac{\mathbf{3.14}}{\mathbf{12}}$[0 +0 +2(3.36)] = **1.7615**

3- Simpson's rule method

Simpson's rule is a numerical method that approximate the value a definite integral by using quadratic polynomial let's derive the formula for the area under equation **(y = ax^2 +bx +c)** passing through three points **(-h,y_0) , (0,y_i), (h,y_2)**



y_0 y_1 y_2

area $= \int_{-h}^{h} (ax2 + bx + c).dx$

$A = \left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right]_{-h}^{h}$

$A = \frac{a}{3}h^3 + \frac{b}{2}h^2 + ch + \frac{a}{3}h^3 - \frac{b}{2}h^2 + ch$

$= \frac{2a}{3}h^3 + 2ch = \frac{h}{3}[2ah^2 + 6c]$ ....(1)

Since the points **$(-h, y_0)$, $(0, y_1)$** and **(h, y_2)**

Are on the equation curve, they satisfy **$y = ax^2 + bx + c$** Therefore

$y_0 = ah^2 - bh + c$

$y_1 = c$

$y_2 = ah^2 + bh + c$

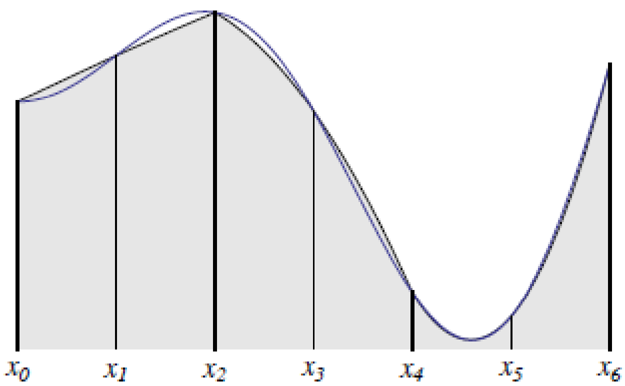
$y_0 + y_1 + y_2 = ah^2 - bh + c + c + ah^2 + bh + c$

$y_0 + y_1 + y_2 = 2ah^2 + 6c$ to similar with equation (1)

$y_0 + 4y_1 + y_2 = ah^2 - bh + c + 4c + ah^2 + bh + c$

$y_0 + 4y_1 + y_2 = [2ah^2 + 6c] = \frac{h}{3}[y_0 + 4y_1 + y_2]$

compute



Area = $\frac{h}{3}$ [y_0 +4 y_1 + y_2] + $\frac{h}{3}$ [y_2 +4 y_3 + y_4] +$\frac{h}{3}$ [y_4 +4 y_5 + y_6]

Area = $\frac{h}{3}$ [y_0 + y_n + $4\sum_{i=1.3.5} yi + 2\sum_{i=2.4.6} yi$]

H =