Why Digital Control Instead of Analogue?

Since almost all control functions can be achieved with analogue (continuous-time) hardware, one is tempted to ask why we might wish to study digital control theory. Engineers have been long interested in the possibility of incorporating digital computers into the control loop because of the ability of a digital computer to process immense quantities of data and base control policies logically on the data. If we follow the historical development of digital computers, we find that they were initially very complex large devices which generally cost too much for application in control systems of a moderate degree of complexity. At this stage cost limitations relegated digital control to only the largest control systems such as those for oil refineries or large chemical-processing plants.

With the introduction of the minicomputer in the mid-1960s the possibilities for digital control were greatly expanded because of reduction of both size and cost. This allowed application of computers to control smaller and less costly systems. The advent of microprocessor unit in the early 1970s has similarly expanded the horizons, since capabilities which formerly cost thousands of dollars may be purchased for, at most, a few hundreds of dollars. These prices make digital control hardware competitive with analogue control hardware for even the simplest single-loop control applications. We now see that microprocessor units complete with system controller, arithmetic unit, clock, limited read only memory (ROM), and random-access memory (RAM) on a single LSI, 40-pin, integrated circuit chip are available for less than five dollars. It is acknowledged that other hardware is required for A/D and D/A conversion but inexpensive hardware is also available to accomplish these tasks. Some of these devices are also beginning to appear aboard the CPU chip. A cost which is not easily estimated is that of software development which is necessary in control applications, but it is safe to say that the more complex the control task the more complex the software required regardless of the digital system employed. Some of these difficulties are now being alleviated by development of high-level languages, such as PASCAL and C-languages, especially suited for microprocessor application. The availability of hardware in LSI form has made digital hardware attractive from space, weight, power consumption, and reliability points of view.

Computational speed is directly affected by hardware speed, and hence there has been considerable effort to increase component speed, which has increased exponentially in the past two decades.

Thus, the development of LSI circuit density which is, in itself, a measurement of the progress in the field of digital hardware. It is interesting to note that the Intel 8086 microprocessor has in excess of 30,000 transistors on a single integrated circuit chip.

Another indicator of speed of technological development is the cost of hardware to accomplish a particular task. If one examines the cost history of a particular line of microprocessors it becomes clear that the price is halved yearly.

The Computer as a Control Unit

Let us consider a single-loop position servomechanism in continuous-time form as shown in Fig.(1). The reference signal is in the form of a voltage, as is the feedback signal, both generated by mechanically driven potentiometers.

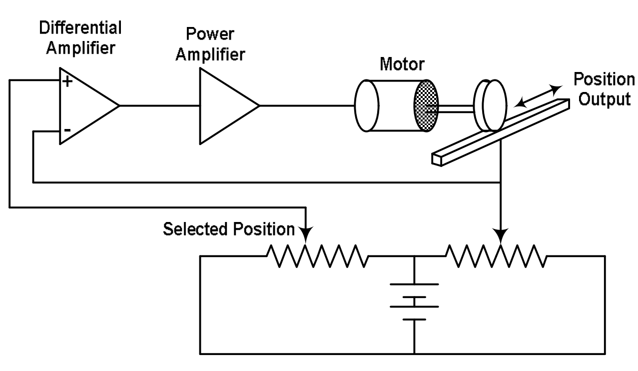


Figure (1) *Position servomechanism with continuous signals*

Let us now investigate how this simple task, outlined in Fig.(1), might be accomplished by employing a digital computer to generate the signal to the power amplifier. We must postulate the existence of two devices. The first of these devices is the analogue-to-digital (A/D) converter which will sample the output signal periodically and convert this sample to a digital word to be processed by the digital computer and thus generate a control strategy in the form of a number. The second device is a digital to analogue (DAC) converter which converts the numerical control strategy generated by the digital computer from a digital word to an analogue signal. The position servomechanism is shown in Fig.(2) controlled by a digital computer.

Generally, the A/D and D/A converters operate periodically and hence the closer together in time the samples are taken, and the more often the output of the D/A converter is updated, the closer the digital control system will approach the continuous-time system. However, it is not always desirable to have the system approach the continuous system in that there are desirable attributes to a discrete-time system.

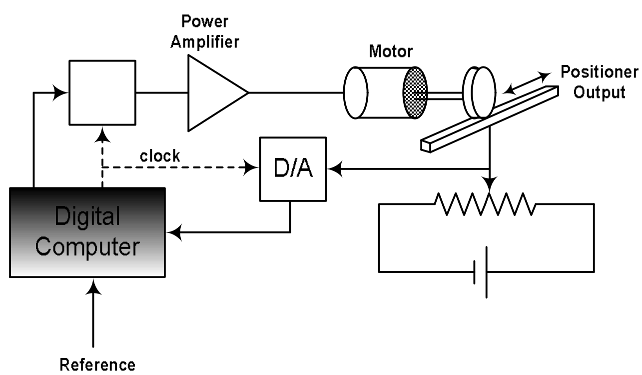
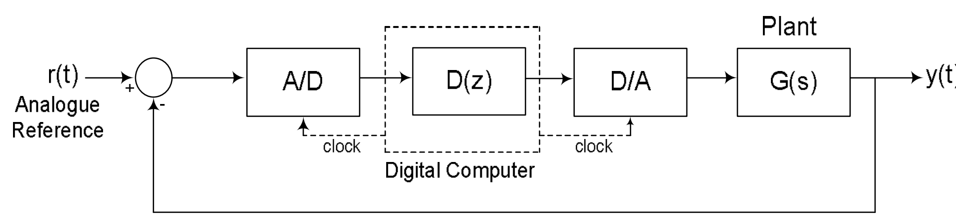


Figure (2) *Digitally controlled positioning system*

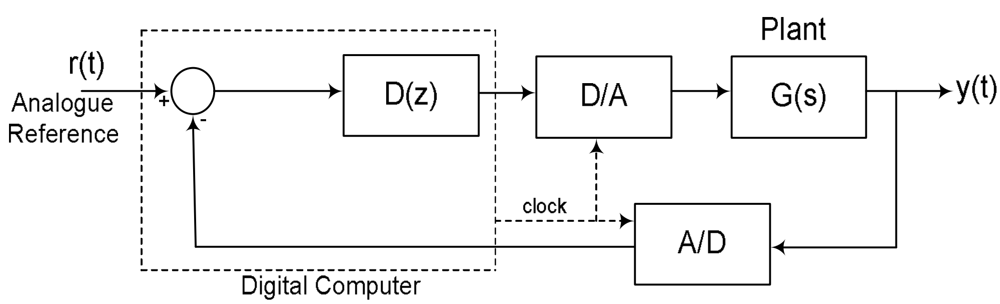
The Single-loop Digital Control System

There are several configurations of a single-loop control system, two of which are shown in Figs.(3-a) and (3-b). In both cases a single continuous-time variable y(t) is being controlled to follow some reference signal which might be zero or constant as in the case of a regulator.

The signal leaving the digital computer in both cases is a periodic sequence of numbers which represent the control strategy as generated by the computer. The input to the digital computer is a periodic sequence of numbers which represent the periodic samples of the continuous signal which is the input to the A/D converter. The purpose of developing digital control theory is to find desirable algorithms by which the digital computer converts the input sequence into the output sequence which is the numerical control strategy. The design process is one of selecting the algorithm reflected in the function D(z).



(a)



(b)

Figure (3) *Several configuration of a digital control system*

Advantages of digital controllers over analogue controllers:

One must recognize that many physical systems have inherent sampling, or their behavior can be described by sampled data or digital models. Many modern control systems contain intentional sampling and digital processors. The sampled-data and digital control are characterized by: *Improved sensitivity, better reliability, no drift, less effect due to noise and disturbance, more compact and light weight, less cost, and more flexibility in programming.*

Some of the advantages of digital controllers over analogue controllers may be summarized as follows:

- Digital controllers are capable of performing complex computations with constant accuracy at high speed. Digital computers can have almost any desired degree of accuracy in computations at relatively little increase in cost. On the other hand, the cost of analogue computers increases rapidly as the complexity of the computations increases if constant accuracy is to be maintained.
- Digital controllers are extremely versatile than analogue controllers. The program which characterizes a digital controller can be modified to accommodate design changes, or adaptive performances, without any variations on the hardware. By merely issuing a new program, one can completely change the operations being performed. This feature is particularly important if the control system is to receive operating information or instructions from some computing center, where economic analysis and optimization studies are being made.
- Because of inability of conventional techniques to adequately handle complex control problems, it has been customary to subdivide a process into smaller units and handle each of these separate control problem. Human operators are normally used to coordinate the operation of units. Recent advances in computer control systems have caused changes in this use of industrial process controls. Recent developments in large-scale

computers and mathematical methods provide a basis for use of all available information in the control system. In conventional control, this part of control loop is being done directly by humans.

- Digital components in the form of electronic parts, transducers and encoders, are often more reliable, more rugged in construction, and more compact in size than their analogue equivalents. These and other glaring comparisons are rapidly converting the control system technology into a digital one.

Z-transform

The simple substitution

$$z = e^{Ts}$$

Converts the Laplace transform to the z transform. Making this substitution into the Laplace transform of the sampled signal

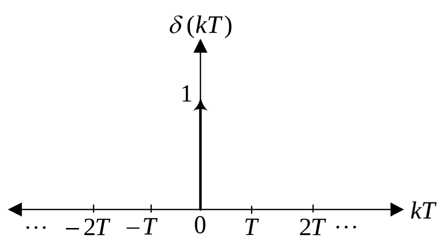
$$\begin{aligned} F^*(s) = Z\ [f^*(t)] = F(z) = &\ f(0) + f(T)\,e^{-Ts} + f(2T)\,e^{-2Ts} + \cdots \\ = &\ f(0) + f(T)\,z^{-1} + f(2T)\,z^{-2} + \cdots \\ = &\sum_{n=0}^{\infty} f(nT)\ z^{-n} \end{aligned} \qquad (1)$$

where F(z) designates the z transform of $f^*(t)$. Because only values of the signal at the sampling instants are considered, the z transform of $f(t)$ is the same as that of $f^*(t)$.

Z Transform by Definition:

In the following analysis, the z transform is derived using Eq.(1), where $f(nT)$ is the function for which the z transform will be obtained.

Impulse function:



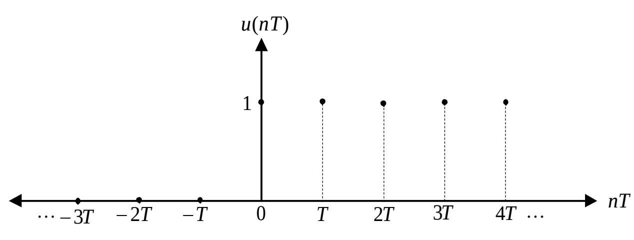
The discrete unit impulse function is defined as

$$\delta(nT) = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$

The z transform of the delta function $\delta(nT)$ can be given as

$$Z\ \{\delta(nT)\} = \sum_{n=0}^{\infty} u_1(nT)\ z^{-n} = z^0 = 1$$

Discrete unit step function:



The discrete step function is defined as

$$u(nT) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

The z transform of the unit step

$$Z\ \{\ u(nT)\ \} = U(z) = \sum_{n=0}^{\infty} u(nT)\ z^{-n} = 1 + z^{-1} + z^{-2} + \ldots.$$

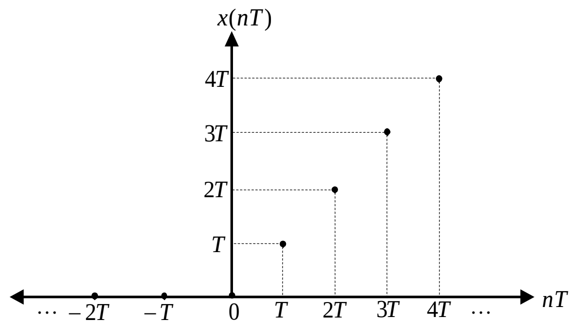
Multiplying both sides of this last equation by z results in

$$z\,U(z) = z + 1 + z^{-1} + z^{-2} + z^{-3} + ... = z + \sum_{n=0}^{\infty} z^{-n} = z + U(z)$$

$$U(z)\ (z-1) = z \quad \text{or} \quad U(z) = \frac{z}{z-1}$$

Discrete ramp function:

The discrete ramp function is defined as



$$x(nT) = \begin{cases} nT & n \geq 0 \\ 0 & n < 0 \end{cases}$$

then

$$X(z) = \sum_{n=0}^{\infty} nT\, z^{-n} = T \sum_{n=0}^{\infty} n\, z^{-n}$$

since $n\,z^{-n} = -z\,\dfrac{d}{dz}\left(z^{-n}\right)$, then

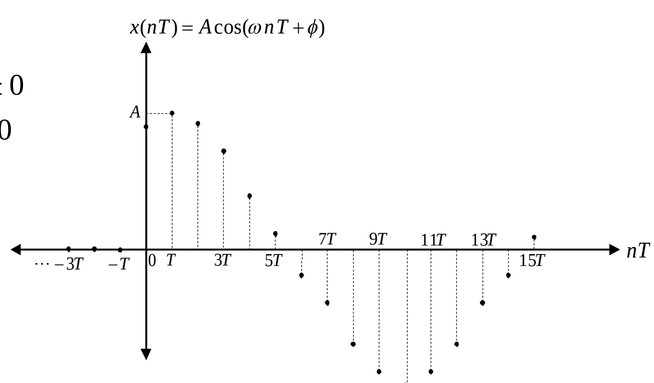
$$X(z) = -Tz\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^{-n}\right)$$

since $\displaystyle\sum_{n=0}^{\infty} z^{-n} = \frac{z}{(z-1)}$, then

Discrete cosine function:

Let

$$x(nT) = \begin{cases} A\cos(\omega nT + \phi) & n \geq 0 \\ 0 & n < 0 \end{cases}$$



The first step is the choice of the alternative representation of cosine function using Euler identity:

Then,

$$X(z) = \frac{A}{2}\sum_{n=0}^{\infty} e^{j\phi} e^{j\omega kT} z^{-n} + \frac{A}{2}\sum_{n=0}^{\infty} e^{-j\phi} e^{-j\omega kT} z^{-n}$$

$$= \frac{A}{2}\frac{z\,e^{j\phi}}{z - e^{-j\omega T}} + \frac{A}{2}\frac{z\,e^{-j\phi}}{z - e^{j\omega}}$$

Discrete exponential decay function:

Let

$$x(nT) = \begin{cases} e^{-anT} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

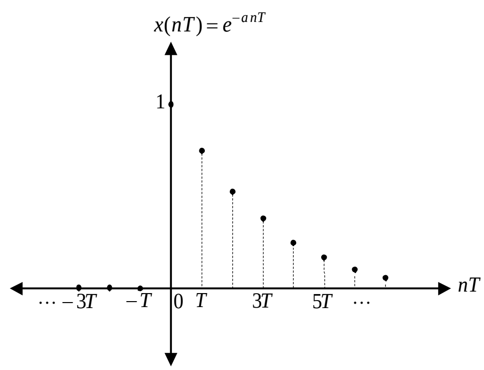
Then,

$$X(z) = \sum_{n=0}^{\infty} e^{-anT} z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{aT} z)^{-n}$$

$$= \frac{(e^{aT} z)}{(e^{aT} z) - 1}$$

$$= \frac{z}{z - e^{-aT}}, \quad |z| > e^{-aT}$$



In Table (1) is given a partial listing of Laplace transforms and corresponding z transforms for commonly encountered functions.

Z Transform Using Partial Fraction:

When the Laplace transform of a function is known, the corresponding z transform may be obtained by the partial fraction

Ex: Determine the z transform for the function whose Laplace transform is

$$F(s) = \frac{1}{s\,(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

From Table (1), the z transform corresponding to $1/s$ is $z/z-1$ and that corresponding to $1/s+1$ is $z/z-e^{-T}$. Thus,

$$F(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z\ (1-e^{-T})}{(z-1)(z-e^{-T})}$$

Table (1) z transforms

Time function	Laplace Transform	Discrete Time function	Z transform
$\delta(t)$	1	$\delta(nT)$	1
$u(t)$	$\frac{1}{s}$	$u(nT)$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	nT	$\frac{zT}{(z-1)^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{(nT)^2}{2}$	$\frac{z(z+1)T^2}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	$e^{-a\,nT}$	$\frac{z}{z-e^{-aT}}$
$t\,e^{-at}$	$\frac{1}{(s+a)^2}$	$nT\,e^{-a\,nT}$	$\frac{zT\,e^{-aT}}{(z-e^{-aT})^2}$
$a^{t/T}$	$\frac{1}{s-(1/T)\ln(a)}$	a^n	$\frac{z}{z-a}\quad (a>0)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega nT)$	$\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\cos(\omega nT)$	$\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$

Ex: Determine the z transform of $\cos(\omega t)$.

It is known that the Laplace transform is $s/(s^2+\omega^2)$. Performing a partial-fraction expansion gives

$$\mathcal{L}\{\cos(\omega t)\}\ \frac{s}{s^2+\omega^2} = \frac{1/2}{s+j\omega} + \frac{1/2}{s-j\omega}$$

The corresponding z transform is

Z Transform Using Residue Method:

This is a powerful technique for obtaining z transforms. The z transform of $f^*(t)$ may be expressed in the form

$$F(z) = Z\,[f^*(t)] = \sum residues\ of\ F(s)\frac{z}{z-e^{sT}}\ at\ poles\ of\ F(s)$$

When the denominator of $F(s)$ contains a linear factor of the form $s-r$ such that $F(s)$ has a first-order pole at $s=r$, the corresponding residue R is

$$R = \underbrace{\lim}_{s\to r}\ (s-r)\left[F(s)\ \frac{z}{z-e^{sT}}\right]$$

When $F(s)$ contains a repeated pole of order q, the residue is

As is illustrated by the following examples, the determination of residues is similar to evaluating the constants in a partial-fraction expansion.

Ex: Determine the z transform of a unit step function.

For $F(s) = 1/s$, there is but one pole at s=0. The corresponding residue is

$$R = \underbrace{\lim}_{s\to 0}\ s\left[\frac{1}{s}\ \frac{z}{z-e^{sT}}\right] = \frac{z}{z-1}$$

Ex: Determine the z transform of e^{-at}.

For this function, $F(s) = 1/(s+a)$, which has but one pole at s=-a. Thus,

$$R = \underbrace{\lim}_{s\to -a}\ (s+a)\left[\frac{1}{(s+a)}\ \frac{z}{z-e^{sT}}\right] = \frac{z}{z-e^{-aT}}$$

Ex: Determine the z transform of for the function whose Laplace transform is $F(s) = \dfrac{1}{s\ (s+1)}$

The poles of $F(s)$ occur at s=0 and s=-1. The residue due to the pole at s=0 is

$$R_1 = \underbrace{\lim}_{s\to 0}\ s\left[\frac{1}{s\ (s+1)}\ \frac{z}{z-e^{sT}}\right] = \frac{z}{z-1}$$

The residue due to the pole at s=-1 is

Adding these two residues results in

$$R = \sum_{i=1}^{2} R_i = R_1 + R_2 = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z\,(1-e^{-T})}{(z-1)(z-e^{-T})}$$

Ex: Determine the z transform of $\cos(\omega t)$.

The Laplace transform is

$$F(s) = \frac{s}{s^2+\omega^2} = \frac{s}{(s-j\omega)(s+j\omega)}$$

The poles are at $s = j\omega$ and $s = -j\omega$. Thus,

Adding these verify the previous result

$$R = \sum_{i=1}^{2} R_i = R_1 + R_2 = \frac{z^2 - z\cos(\omega T)}{z^2 - 2z\cos(\omega T) + 1}$$

Ex: Determine the z transform corresponding to the function $f(t) = t$.

The Laplace transform is

$$F(s) = \frac{1}{s^2}$$

This has a second-order pole at s=0. Thus, the residue becomes

$$R = \frac{1}{(2-1)!} \underbrace{\lim}_{s\to 0} \frac{d^{2-1}}{ds^{2-1}} \left[(s-r)^2 \; F(s) \; \frac{z}{z-e^{sT}} \right]$$

or

Theorems

- Initial value theorem:

Suppose $f(nT)$ has z transform $F(z)$ and $\lim_{z\to\infty} F(z)$ exist, then the initial value $f(0)$ of $f(nT)$ is given by

Proof: Note that

$$F(z)=\sum_{n=0}^{\infty} f(nT)\ z^{-n} = f(0)+f(T)\frac{1}{z}+f(2T)\frac{1}{z^2}+\cdots$$

letting $z\to\infty$, the theorem is verified.

- Final value theorem :

Suppose $f(nT)$ has z transform F(z). Then,

Proof: Consider the following sums S_n and S_{n-1}

Dividing the second series by *z* and then subtracting the second from the first gives

$$\left(1-\frac{1}{z}\right)f(0)+\cdots+\left(1-\frac{1}{z}\right)\frac{f[(n-1)T]}{z^{n-1}}+\frac{f(nT)}{z^n}$$

Taking the limit as z approaches 1 gives

When *n* is very large, $S_{n-1}\approx S_n\approx F(z)$. Thus, the final-value theorem given by Eq.(1) is verified.

Ex: For a discrete data system with transfer function

$$H(z)=\frac{Y(z)}{U(z)}=\frac{z+1}{z^2-1.4z+0.48}$$

and a unit step input for which the z transform is

$$U(z)=\frac{z}{z-1}$$

find the final value of the response sequence $y(nT)$. The response in the z domain is

$$Y(z)=\frac{z\,(z+1)}{(z^2-1.4z+0.48)(z-1)}$$

and employing the final value theorem

$$y(\infty)=\lim_{z\to 1}\left[\frac{z-1}{z}\,Y(z)\right]$$

- <u>Shifting property:</u>

<u>Left shifting</u>: When $f(nT)$ is delayed k sampling instants, the function $f(nT-kT)$ shown in Fig.(1-b) results. The value of $f(nT-kT)$ when $n=k$ is $f(0)$, the value when $n=k+1$ is $f(T)$, etc. The z transform of $f(nT-kT)$ is

$$\begin{aligned} Z[f(nT-kT)&=\sum_{n=0}^{\infty} f(nT-kT)\,z^{-n} \\ &=\underbrace{f(-kT)}_{n=0}+\underbrace{f(1-kT)}_{n=1}+\cdots+\underbrace{f(0)\,z^{-k}}_{n=k}+\underbrace{f(1)\,z^{-(k+1)}}_{n=k+1}+\underbrace{f(2)\,z^{-(k+2)}}_{n=k+2}+\cdots \\ &=f(0)z^{-k}+f(1)z^{-(k+1)}+f(2)z^{-(k+2)}+\cdots \\ &=z^{-k}\,(f(0)+f(1)z^{-1}+f(2)z^{-2}+\cdots) \\ &=z^{-k}\,F(z) \end{aligned}$$

$\therefore$

<u>Right shifting</u>: When the function $f(nT)$ of Fig.(1-a) is shifted one sampling period to the left, the function $f(nT+T)$ shown in Fig.(1-c) results. The value of $f(nT+T)$ when $n=0$ is $f(T)$, the value when $n=1$ is $f(2T)$, etc. The z transform of $f(nT+T)$ is

$$Z[f(nT+T)=\sum_{n=0}^{\infty} f(nT+T)\,z^{-n}=f(T)+f(2T)z^{-1}+f(3T)z^{-2}+\cdots$$

Multiplying through both sides by z^{-1} and adding $f(0)$ to both sides gives

$$z^{-1}\,Z[f(nT+T)]+f(0)=f(0)+f(T)\,z^{-1}+f(2T)\,z^{-2}+\cdots=F(z)$$

Thus,

$$Z[f(nT+T)] = z\,F(z) - z\,f(0)$$

Similarly, it follows that

$$Z[f(nT+2T)] = z^2\,F(z) - z^2\,f(0) - z\,f(1)$$

In general,

- Multiplication by (nT):

  The z transform of $nT\,f(nT)$ is

Proof: To verify this theorem, note that

$$\begin{aligned} -T\,z\frac{d}{dz}F(z) &= -T\,z\frac{d}{dz}[f(0) + f(T)\,z^{-1} + f(2T)\,z^{-2} + \cdots] \\ &= T[f(T)\,z^{-1} + 2\,f(2T)\,z^{-2} + 3\,f(3T)\,z^{-3} + \cdots] \\ &= T[f(T)\,z^{-1} + 2\,f(2T)\,z^{-2} + 3\,f(3T)\,z^{-3} + \cdots] \\ &= \sum_{n=0}^{\infty} nT\;f(nT)\,z^{-nT} \\ &= Z[nT\;f(nT)] \end{aligned}$$

- Multiplication by a^n:

  The z transform of $a^n\;f(nT)$ is

  [redacted]

Proof: This theorem is readily proved by placing z by z/a in F(z)

$$F(z/a) = f(0) + \frac{a\,f(T)}{z} + \frac{a\,f(2T)}{z^2} + \cdots$$

$$F(z/a) = f(0) + f(T)\left(\frac{z}{a}\right)^{-1} + f(2T)\left(\frac{z}{a}\right)^{-2} + \cdots = \sum_{n=0}^{\infty} f(nT)\left(\frac{z}{a}\right)^{-n}$$

The right side is the z transform of $a^n\;f(nT)$.

Ex: The z transform of a unit step function,

$$f(nT)=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$$

is given by $F(z)=\dfrac{z}{z-1}$, then

$$Z[a^n]=F(z/a)=\frac{(z/a)}{(z/a)-1}=\frac{z}{z-a}$$

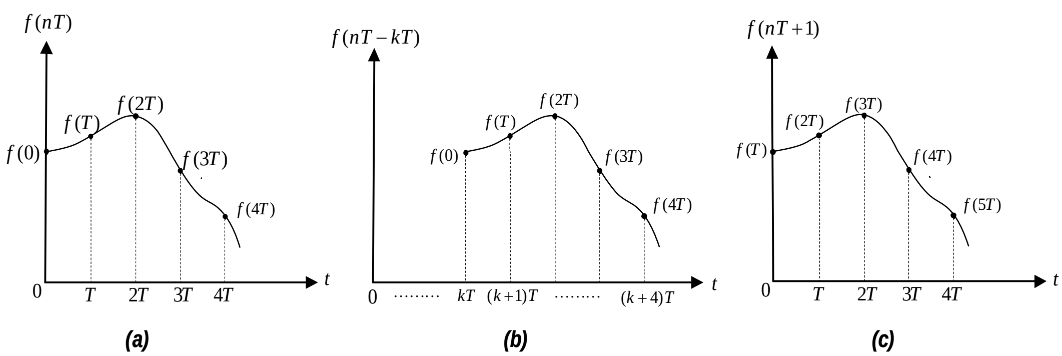


Figure (1) *Translation of a discrete function* $f(nT)$

A listing of z transform theorems and properties is given in Table(2)

Table (2) Properties of the z transforms

$f(nT)$	$Z\,[f(nT)]$
$a\ f(nT)$	$a\ F(z)$
$f_1(nT)+f_2(nT)$	$F_1(z)+F_2(z)$
$f(kT-nT)$	$z^{-n}\ F(z)$
$f(kT+T)$	$z\,F(z)-z\ f(0)$
$f(kT+2T)$	$z^2\ F(z)-z^2\ f(0)-z\ f(T)$
$f(kT+nT)$	$z^n\ F(z)-z^n\ f(0)-z^{n-1}\ f(T)-\cdots-z\ f[(n-1)T]$
$nT\ f(nT)$	$-zT\dfrac{d}{dz}(F(z))$
$e^{-a\,(nT)}\ f(nT)$	$F\left(z\,e^{aT}\right)$
$a^{(nT)}\ f(nT)$	$F\left(z/a\right)$
$\dfrac{\partial}{\partial a}\ f(nT,a)$	$\dfrac{\partial}{\partial z}\ F\left(z/a\right)$

Inverse z transform

Inspection of Table (1) shows that z transform tend to be more complicated than corresponding Laplace transforms. Fortunately, there are some relatively simple techniques for obtaining inverse z transforms.

- Partial-Fraction Method:

In this method, obtaining $x(nT)$ is based on the partial fraction expansion of $X(z)/z$ and the identification of each of the terms by the use of a table of z transforms. Note that the reason we expansion we $X(z)/z$ into partial fractions is that the functions of z appearing in tables of z transforms usually have the factor z in their numerators.

Consider $X(z)$ given by

$$X(z)=\frac{b_o z^m + b_1 z^{m-1} + \cdots + b_{m-1}z + b_m}{a_o z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n} \qquad (m \le n)$$

- ☐ Factor the denominator polynomial of $X(z)$ and find the poles.
- ☐ Expand $X(z)/z$ into partial fractions so that each of the terms is easily recognizable in a table of z transforms.

Ex: Find $x(nT)$ if $X(z)$ is given by

$$X(z)=\frac{10\,z}{(z-1)(z-2)}$$

we first expand $X(z)/z$ into partial fractions as follows

Then we obtain

$$X(z)=\frac{-10\ z}{z-1}+\frac{10\ z}{z-2}$$

From table (1), one can obtain

$$Z^{-1}\left[\frac{z}{z-1}\right]=1, \qquad Z^{-1}\left[\frac{z}{z-2}\right]=2^n$$

Hence

$$x(nT)=10\,(-1+2^n) \qquad n=0,\ 1,\ 2,\ \ldots$$

Ex: Find $f(nT)$ if $F(z)$ is given by

$$F(z)=\frac{(1-e^{-T})\,z}{(z-1)(z-e^{-T})}$$

Performing a partial fraction expansion of $F(z)/z$ gives

$$\frac{F(z)}{z}=\left[\frac{(1-e^{-T})}{(z-1)(z-e^{-T})}\right]=\left[\frac{K_1}{(z-1)}+\frac{K_2}{(z-e^{-T})}\right]$$

or

$$F(z)=\left[\frac{z}{(z-1)}-\frac{z}{(z-e^{-T})}\right]$$

From Table (1), the corresponding discrete time function

$$f(nT)=1-e^{-nT} \qquad\qquad n=0,\ 1,\ 2,\ \ldots$$

- Residue Method:

The third method of finding the inverse z transform is to use the inversion integral. Note that

$$F(z)=\sum_{n=0}^{\infty} f(nT)\, z^{-n}=f(0)+f(T)z^{-1}+f(2T)z^{-2}+\cdots$$

By multiplying both sides of this last equation by z^{n-1}, we obtain

$$F(z)z^{n-1} = f(0)z^{n-1} + f(T)\,z^{n-2} + f(2T)\,z^{n-3} + \cdots \qquad (*)$$

If $s=\sigma+j\omega$ is substituted in this last equation, we obtain $z=e^{(\sigma+j\omega)T}$, or

$$|\,z\,|=e^{\sigma T}, \qquad z \neq \omega T$$

If the poles of **L**[*x*] lie to the left of the line $s=\sigma_1$ in the s plane, the poles of **Z** [x] will lie inside the circle with its center at the origin and radius equal to $e^{\sigma_1 T}$ in the z plane.

Suppose we integrate both sides of Eq.$(*)$ along this circle in the counterclockwise direction:

$$\oint F(z)z^{n-1}dz=\oint f(0)z^{n-1}dz+\oint f(T)\,z^{n-2}dz+\cdots+\oint f(T)\,z^{-1}dz+\cdots$$

Applying Cauchy's theorem, we see that all terms on the right-hand side of this last equation are zero except one term

$$\oint f(T)\,z^{-1}dz$$

Hence

$$\oint F(z)z^{n-1}dz=\oint f(T)\,z^{-1}dz$$

from which we obtain the inversion integral for the z transform

$$x(nT)=\frac{1}{2\pi j}\oint X(z)\ z^{k-1}\,dz$$

which is equivalent to stating that

$$x(nT) = \sum [\, residues\ of\ X(z)\, z^{n-1}\ at\ poles\, of\ X(z)\,]$$

In particular, the residue due to a first order pole at $z = r$ is

$$R = \lim_{z \to r} (z - r)\ [F(z)\, z^{n-1}]$$

Similarly, the residue due to a repeated pole of order q is

$$R = \frac{1}{(q-1)!} \lim_{z \to r} \frac{d^{q-1}}{dz^{q-1}} [(z-r)^q\ F(z)\, z^{n-1}]$$

<u>Ex:</u> Using residue method, find $f(nT)$ if $F(z)$ is given by

$$F(z) = \frac{(1 - e^{-T})\, z}{(z-1)(z - e^{-T})}$$

Application of the residue method to determine the inverse of the above equation

Adding these residues gives

$$f(nT) = 1 - e^{-nT} \qquad n = 0, 1, 2, 3, \ \ldots$$

<u>Ex:</u> Determine the inverse z transform for the function

$$F(z) = \frac{Tz}{(z-1)^2}$$

This function has a second-order pole at $z = 1$; thus

For $f(nT) = nT$, the corresponding time function is $f(t) = t$.

Sampling Theory

The Unit-impulse Train

Let us first consider the Dirac delta or unit-impulse function located at t=a as shown in Fig.(1). The delta function will be denoted as $\delta(t-a)$ and it will be defined by the relations

$$\delta(t-a) = \begin{cases} 1 & t = a \\ 0 & t \neq a \end{cases} \qquad (1)$$

and

$$\int_{-\infty}^{\infty} \delta(t-a)\ dt = 1 \qquad (2)$$

One should first state the shifting or sampling property of the impulse function which is given by

$$f(a) = \int_{-\infty}^{\infty} f(t)\ \delta(t-a)\, dt \qquad (3)$$

Let us define a periodic train of unit-impulse function $\delta(t-a)$ as illustrated in Fig.(2). This so-called function can written as a series of Dirac delta functions or

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) \qquad (4)$$

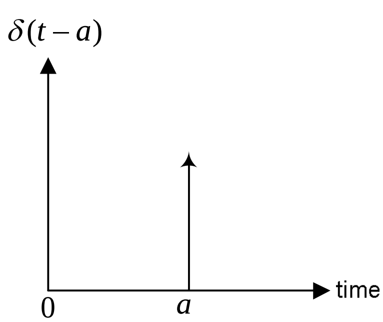


Figure (1) *Dirac delta or unit impulse function*

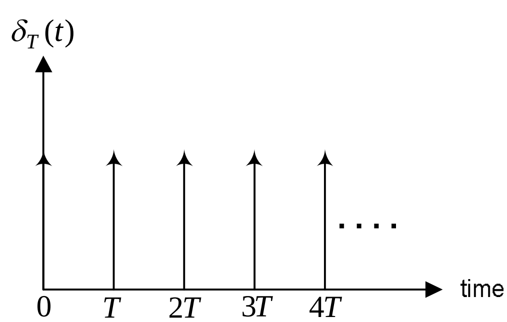


Figure (2) *Unit impulse train*

Since this train of impulse functions is a periodic function with a period T and fundamental radian frequency $\omega_s = 2\pi / T$, it is reasonable to discuss the complex Fourier series of this periodic function or

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} \qquad (5)$$

where the complex Fourier series coefficients are given by

(6)

Since the interval of integration includes only the impulse function at the origin, one can employ the shifting property of that impulse to yield

(7)

which implies that the Fourier series representation of $\delta_T(t)$ is non-convergent. This tells us that the contribution of each frequency to the waveform is equal. The infinite train of impulse may be written as

(8)

The Impulse Sampling Model

Let us generate the sampled version $f^*(t)$ of some arbitrary function $f(t)$ to amplitude-modulate the impulse train $\delta_T(t)$. This is simply done by multiplying the two functions together as shown in Fig.(3). Then, it is clear that $f^*(t)$ is given by

$$f^*(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) \qquad (9)$$

It is assumed here that there were no jumps in the function $f(t)$ at the sampling instants. Let us now consider temporal functions which are Laplace transformable (i.e., functions which are zero negative time and of less than exponential order), where the transform is defined in the usual sense by

$$F(s) = £\ [\ f(t)\ ] = \qquad (10)$$

If expression (8) for the impulse train has been substituted in Eq.(9), one can get

(11)

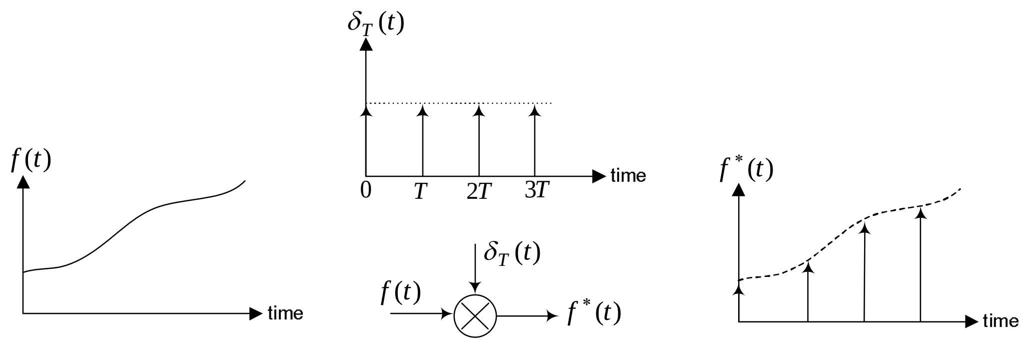


Figure (3) *Impulse train modulator*

Now let us find the Laplace transform of $f^*(t)$ and call it $F^*(s)$ by using the definition of the Laplace transform

(12)

In the light of the definition of $F(s)$ given by expression (10), one can see that the terms under the summation are the same except for the arguments being shifted by $jn\omega_s$ so

(13)

This expression states that the operation of impulse sampling $f(t)$ has made the Laplace transform of $f^*(t)$ periodic in the s domain.

The Samping Theory

The implications of expression (13) are wide and sweeping, but one of the most interesting is the sampling theorem. Let us consider only-signals $f(t)$ which have Fourier transform $F(j\omega)$. The operation of sampling gives a sampled frequency domain function which can be evaluated by simply evaluating the expression (13) on the $j\omega$ axis, or

$$F^*(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(j\omega - jn\omega_s) \tag{14}$$

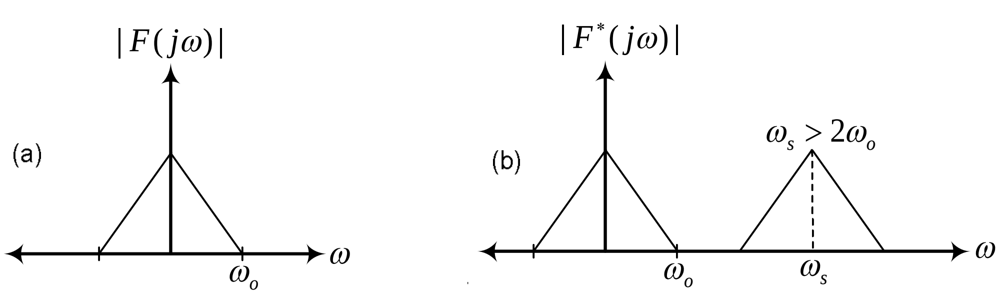
where $F(j\omega)$ is the Fourier transform of the unsampled function $f(t)$. This expression states that portions of $F(j\omega)$ which existed for $\omega > \omega_s/2$ are now mapped down into the primary frequency band $-\omega_s/2 < \omega < \omega_s/2$ and from the sampled function it is impossible to separate these contributions from those which came from that band in the unsampled function. This is the dilemma of sampling in that by sampling, information about the signal is lost which can never be retrieved unless the signal has no frequency content greater than $\omega_s/2$.

In short, it is best to now state the sampling theorem as first advanced by Nyquist (1928) and later proven from an information theoretic point of view by Shannon (1948). The sampling theorem states:

If a signal contains no frequency components above ω_o rad/sec, then it is completely described by its sampled values uniformly spaced in time with period $T < (\pi/\omega_b)$ sec. The signal can be reconstructed from a sampled waveform $f^*(t)$ by passing it through an ideal low-pass filter with bandwidth ω_f where $\omega_o < \omega_f < \omega_s - \omega_o$ with $\omega_s = 2\pi/T$. The frequency ω_o is referred to as the "Nyquist frequency."

<u>Example 1</u>:

Consider a function which has a magnitude spectrum ($|F(j\omega)|$) as indicated in Fig.(4). This signal has a cutoff frequency of ω_o rad/sec and we shall consider sampling this signal at different rates denoted by sampling frequency ω_s.



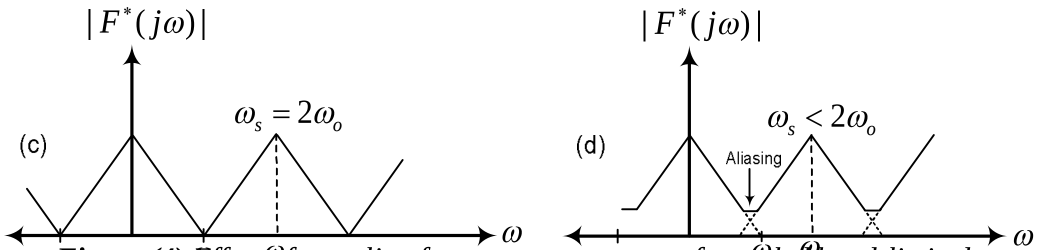


***Figure (4)** Effect of sampling frequency on spectra of sampled band-limited*

- When $\omega_s > 2\omega_o$, the expression (14) indicates that the spectrum will be periodic, but since $\omega_s > 2\omega_o$, there will be no overlapping so we get the spectrum of Fig.(4-b).
- If the sampling frequency has been decreased to exactly twice the cutoff frequency of the signal, then we get a spectrum as indicated in Fig.(4-c) still with no overlapping.
- If the sampling frequency is lowered such that $\omega_s < 2\omega_o$, now the contributions of adjacent terms are additive in a band of frequencies around the frequency $\omega_s / 2$ as illustrated in the spectrum of Fig.(4-d). We see that by sampling at too low a rate some of the lower end of the second lobe of the periodic spectrum has crept down into the primary frequency band $-\omega_s / 2 < \omega < \omega_s / 2$. There is no way to process the resulting sampled data to get rid of this contribution, which is called "aliasing" or "folding."

Example 2:

Consider the following function of time:

$$f(t) = \begin{cases} 0 & t < 0 \\ 5e^{-t}\sin(3t) - 3e^{-2t} & t \geq 0 \end{cases}$$

The Laplace transform of this function is

$$F(s) = \frac{-3s(s-3)}{(s+2)(s+1)^2 + 3^2}$$

for which the pole-zero plot is shown in Fig.(5-a). Now if we sample $f(t)$ to create $f^*(t)$ at a rate of $\omega_s = 4$ rad/sec, we see that the complex pair of poles lie outside the primary strip in the s-plane, indicating that we did not sample fast enough to get accurate information on the frequency content of $f(t)$. If we investigate the implication of the expression (13), we see that the complex poles at $(-1\,,\pm\,j3)$ are now mapped down into the primary strip at $(-1\,,\pm\,j1)$ which creates frequency content in $f^*(t)$ which was not present in $f(t)$. The pole-zero plot for $f^*(t)$ is shown in Fig.(5-b).

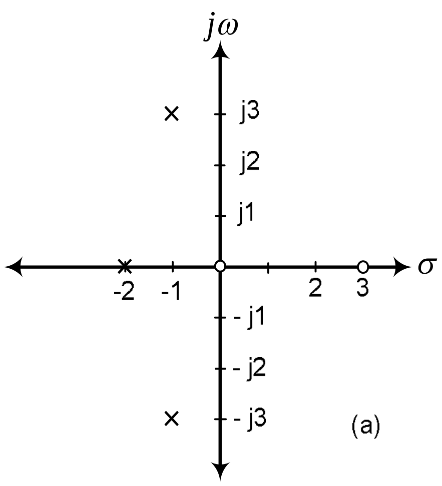
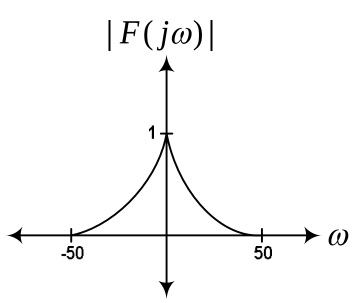


Figure (5) *Pole-zero plots for a function and its sampled version: (a) pole-zero plot of F(s); pole-zero plot of F(s) with* $\omega_s = 4$ *rad/sec.*

HW1. For a function with the Fourier magnitude spectrum shown in Fig.(6), sketch the Fourier spectrum of the sampled version of the waveform for sampling periods of (a) $2\pi/100$ (b) $2\pi/200$.

HW2. A waveform has a Fourier magnitude spectrum illustrated In Fig.(8). Sketch the spectra of the sampled function if it is sampled at frequencies of (a) 800 rad /sec; (b) 1600 rad/sec.



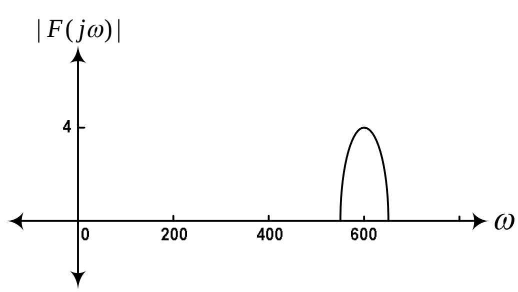


Figure (6) *Figure (7)*

HW3. A periodic waveform with a fundamental frequency of 200 Hz has the Fourier magnitude spectrum illustrated in Fig.(8). Sketch the resultant spectrum if the waveform is sampled at frequencies (a) ll00 Hz; (b) 1000 Hz.

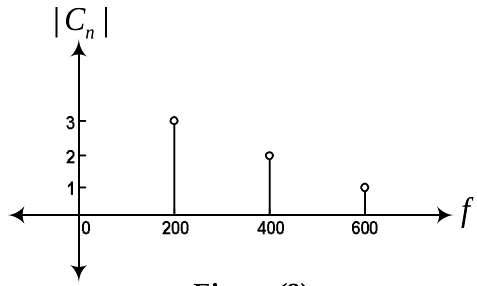


Figure (8)

HW4. Sketch the pole-zero diagram for the sampled versions of the time domain functions associated with s-domain functions given below for a sampling frequency ω_s =10 sec.

a) $F(s) = \dfrac{5}{s+5}$ b) $G(s)\dfrac{s+2}{s^2+4s+40}$ c) $H(s) = \dfrac{s^2+4s+8}{s^2+4s+20}$

Pulse Transfer Function

In Fig.(1) is shown a sampling switch followed by a linear element whose transfer function is G(s). The transformed equation for the output Y(s) is

$$Y(s) = F^*(s)G(s) \tag{1}$$

- For $0 < t < T$, the response y(t) is that due to the first impulse at $t = 0$ of area *f(0)*. Thus, for this interval

$$y(t) = \mathcal{L}^{-1}[f(0)\,\mathrm{G(s)}] = f(0)\,\mathcal{L}^{-1}[\mathrm{G(s)}] = f(0)g(t)$$

where *g(t)*= $\mathcal{L}^{-1}$[G(s)] is the response of the linear element to a unit impulse which occurs at t=0.

- For $T < t < 2T$, the response y(t) is that due to the first impulse at $t = 0$ plus that at $t = T$. For this interval, $F^*(s) = f(0) + f(T)e^{-Ts}$. Thus,

$$Y(s) = [\,f(0) + f(T)e^{-Ts}\,]\,G(s)$$

Inverting gives

$$y(t) = f(0)g(t) + f(T)g(t-T)$$

where *g(t-T)*= $\mathcal{L}^{-1}$[G(s)e^{-Ts}] is the response of the linear element to a unit impulse which occurs at t=T.

- For $2T < t < 3T$, the response y(t) becomes

$$y(t) = f(0)g(t) + f(T)g(t-T) + f(2T)g(t-2T)$$

In general, the response $y(t)$ is

$$y(t) = \sum_{n=0}^{\infty} f(nT)g(t-T) \tag{2}$$

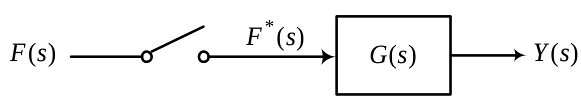


Figure (1) *A sampling switch followed by a linear element*

When n is such that $nT > t$, then $g(t-nT)$ is zero. That is, the impulse response is zero for negative time. For $t = 0, T, 2T, \ldots$, etc, Eq.(2) becomes

$$\left.\begin{aligned} y(0) &= f(0)g(0) + f(T)\underbrace{g(-T)}_{=0} + f(2T)\underbrace{g(-2T)}_{=0} + \cdots \\ y(T) &= f(0)g(T) + f(T)g(0) + f(2T)\underbrace{g(-T)}_{=0} + f(3T)\underbrace{g(-2T)}_{=0} + \cdots \\ y(2T) &= f(0)g(2T) + f(T)g(T) + f(2T)g(0) + f(3T)\underbrace{g(-T)}_{=0} + f(3T)\underbrace{g(-2T)}_{=0} + \cdots \end{aligned}\right\} \quad (3)$$

One can classify the response $y(t)$ depending on the order of denominator with respect to that of numerator:

- When the order of denominator exceeds the order of numerator by only one, the response function $y(t)$ becomes discontinuous at the sampling instants, as shown in Fig.(2). Then, Eq.(1) yields the values of $y(t)$ immediately after the sampling instants [that is, $y(0+)$, $y(T+)$, $y(2T+), \ldots$].
- When the order of denominator exceeds the order of numerator by two or more, $y(t)$ is continuous at the sampling instants. Then, Eq.(1) yields the values at the sampling instants.

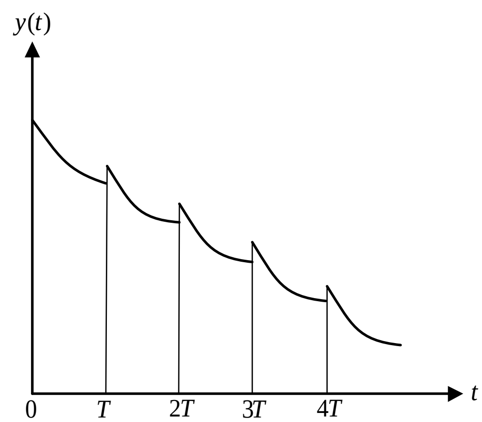


Figure (2) *Response function which is discontinuous at the sampling instant*

Sampling of $Y(s)$ gives

$$Y^*(s) = y(0) + y(T)e^{-Ts} + y(2T)e^{-2Ts} + \cdots$$

Substitution of the values from Eq.(1) into the preceding expression gives:

$$Y^*(s) = f(0)\,g(0) + [\,f(0)\,g(T) + f(T)\,g(0)]e^{-Ts} + \cdots$$
$$= f(0)[g(0) + g(T)e^{-Ts} + g(2T)e^{-2T} + \cdots]$$
$$+ f(T)e^{-Ts}[g(0) + g(T)e^{-Ts} + g(2T)e^{-2T} + \cdots]$$
$$+ \cdots\cdots$$

$$Y^*(s) = [\,f(0) + f(T)e^{-Ts} + f(T)e^{-2Ts} + \cdots][g(0) + g(T)e^{-Ts} + g(2T)e^{-2T} + \cdots]$$

Thus,

$$Y(s) = F^*(s)G^*(s) \qquad (4)$$

The term $G^*(s)$ is called the pulse-transfer function.

Comparison of Eq.(1) and Eq.(4) reveals a basic mathematical relationship for starring quantities. That is, starring both sides of Eq.(1) gives

[redacted]

[redacted]

Letting $z = e^{Ts}$ in Eq.(4) yields the z transform relationship

$$Y(z) = F(z)G(z)$$

Ex: For the sampler configuration shown in Fig.(3), determine the pulse transfer function when $G_1(s) = (1/s)$ and $G_2(s) = 1/(s+1)$

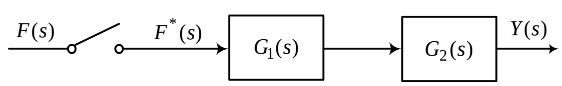


Figure (3) *Sampled-data system*

The Laplace transform relationship is

$$Y(s) = F^*(s)G_1(s)G_2(s)$$

Starring gives

$$Y^*(s) = F^*(s)[G_1(s)G_2(s)]^* = F^*(s)G_1G_2{}^*(s)$$

where $\quad [G_1(s)G_2(s)]^* = G_1G_2{}^*(s)$

The corresponding z transform is

$$Y(z) = F(z)G_1G_2(z)$$

The pulse transfer function becomes

$$\frac{Y(z)}{F(z)} = G_1G_2(z)$$

The product $G_1(s)G_2(s)$

$$G_1(s)G_2(s) = \frac{1}{s(s+1)}$$

The z transform for this function is given by

$$G_1\,G_2(z) = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

<u>Ex:</u> For the sampler configuration shown in Fig.(4), determine the pulse transfer function when $G_1(s) = (1/s)$ and $G_2(s) = 1/(s+1)$

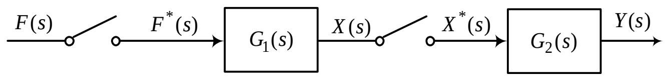


Figure (4) *Sampled-data system*

The Laplace transform relationship is

$$X(s) = F^*(s)G_1(s)$$
$$Y(s) = X^*(s)G_2(s)$$

Starring the first equation and then substituting this result for $X^*(s)$ into the second equation gives

$$Y(s) = F^*(s)G_1^*(s)G_2(s)$$

starring gives

$$Y^*(s) = F^*(s)G_1^*(s)G_2^*(s)$$

The corresponding z transform is

$$Y(z) = F(z)G_1(z)G_2(z)$$

Then the pulse transfer function becomes

$$\frac{Y(z)}{F(z)} = G_1(z)G_2(z)$$

The z transforms of $G_1(s)$ and $G_2(s)$ are

$$G_1(z) = \frac{z}{z-1}\ ,\quad G_2(z) = \frac{z}{z-e^{-T}}$$

Thus

$$G_1G_2(z) = \frac{z^2}{(z-1)(z-e^{-T})}$$

From the preceding two examples it is to be noted that

[redacted]

Ex: For the two sampled-data feedback systems, find the pulse transfer function.

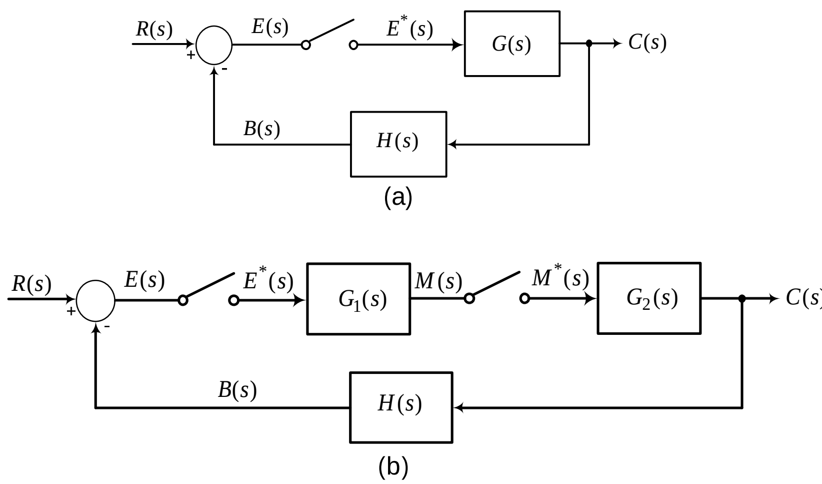


Figure (5) *Sampled-data system*

The equation relating the inputs and outputs of Fig.(5-a) are

$$C(s) = E^*(s)G(s)$$
$$E(s) = R(s) - E^*(s)G(s)H(s)$$

Starring gives

$$C^*(s) = E^*(s)G^*(s)$$
$$E^*(s) = R^*(s) - E^*(s)GH^*(s)$$

Solving the last equation for $E^*(s)$ and substituting into the first gives

$$C^*(s) = \frac{G^*(s)}{1+GH^*(s)}R^*(s)$$

The corresponding z transform is

$$C(z) = \frac{G(z)}{1+GH(z)}R(z)$$

The equation relating the inputs and outputs of Fig.(5-b) are

$$C(s) = M^*(s)G_2^*(s)$$
$$M(s) = E^*(s)G_1(s)$$
$$E(s) = R(s) - M^*(s)G_2(s)H(s)$$

Starring all equations, then solving for $C^*(s)$ gives

$$C^*(s) = \frac{G_1^*(s)G_2^*(s)}{1 + G_1^*(s)G_2H^*(s)}R^*(s)$$

The corresponding z transform is

$$\frac{C(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1 + G_1(z)G_2H(z)}R(z)$$

Zero Order Hold

Digital-to-Analogue Converter:

The digital-to-analogue converter is the device which converts the numerical contect of some register of the digital processor to an analoge voltage and holds the voltage constant until the content of the register is update, and then the output of the digital-to-analogue converter is updated and held again. The D/A converter will be modelled as a zero order hold.

Filters and Zero-Order Hold:

Sampled-data systems usually incorporate a filter, as illustrated in Fig.(1). A perfect filter would convert the sampled data signal $f^*(t)$ back to the continuous input $f(t)$. That is, the output $y(t)$ of the filter would equal $f(t)$. If such a perfect filter were possible, then the sampled-data system would behave the same as the continuous system.

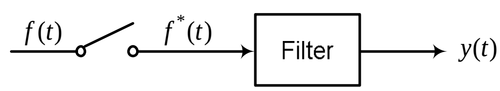


Figure (1) *Schematic representation of sampler and filter*

The most commonly used filter is that in which the value of the last sample is retained until the next sample is taken. This type of filter is called a zero-order hold. Figure (2) shows the operation of zero-order hold. The continuous curve represents the continuous function $f(t)$. The discrete lines are values of $f(t)$ at the sampling instants (the sampled signal $f^*(t)$). Because the zero-order hold retains the value of $f(t)$ at each sampling instants, $y(t)$ is the series of steps.

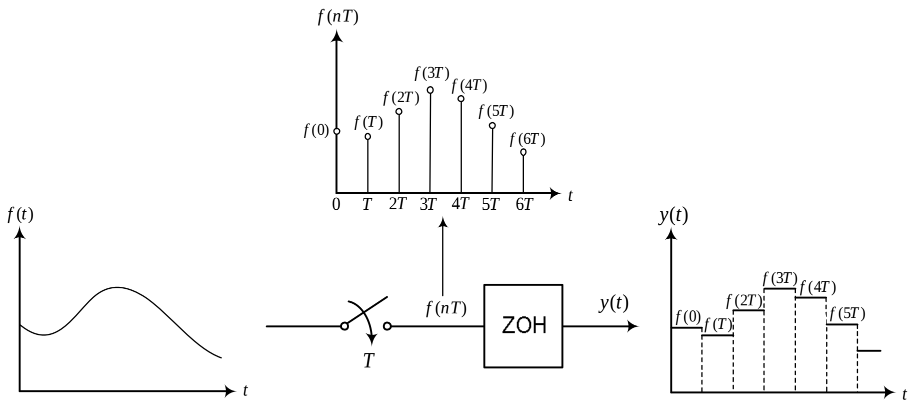


Figure (2) *Zero order hold operation*

The signal at the output of the zero order hold $y(t)$ could be decomposed into a series of pulse functions as shown in Fig.(3). Therefore, one can write the output signal of the ZOH as follows:

$$y(t) = u_o + u_1 + u_2 + \cdots \tag{1}$$

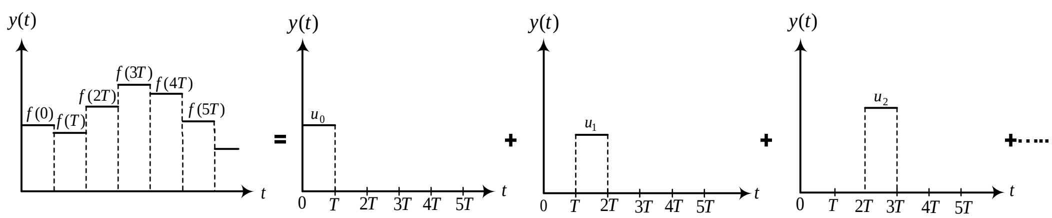


Figure (3) *Decomposition of the zero-order hold output*

The pulse functions u_0 and u_1 of Fig.(3) could be further decomposed into step functions as shown in Fig.(4). One can write u_0 and u_1 as follows

$$u_0 = f(0)[u(t) - u(t-T)] \quad , \quad u_1 = f(T)[u(t-T) - u(t-2T)]$$

where $u(t)$ is unit step function and $u(t-T)$ is unit step function delayed by one sampling time T, etc.Thus, Eq.(1) can be rewritten as

$$f(t) = f(0)[u(t) - u(t-T)] + f(T)[u(t-T) - u(t-2T)] \\ + f(2T)[u(t-2T) - u(t-3T)] + \cdots$$

Taking the Laplace transform of the preceding equation:

$$Y(s) = f(0)\left(\frac{1-e^{-Ts}}{s}\right) + f(T)\left(\frac{e^{-Ts} - e^{-2Ts}}{s}\right) + f(2T)\left(\frac{e^{-2Ts} - e^{-3Ts}}{s}\right) + \cdots$$

$$Y(s) = \left(\frac{1 - e^{-Ts}}{s}\right) [f(0) + f(T)e^{-Ts} + f(2T)e^{-2Ts} + f(3T)e^{-2Ts} + \cdots]$$

$$Y(s) = \left(\frac{1 - e^{-Ts}}{s}\right) F^*(s)$$

$$Y(s) = G_{zoh}(s)\ F^*(s) \qquad (2)$$

where $G_{zoh}(s)$ is the transfer function of zero order hold.

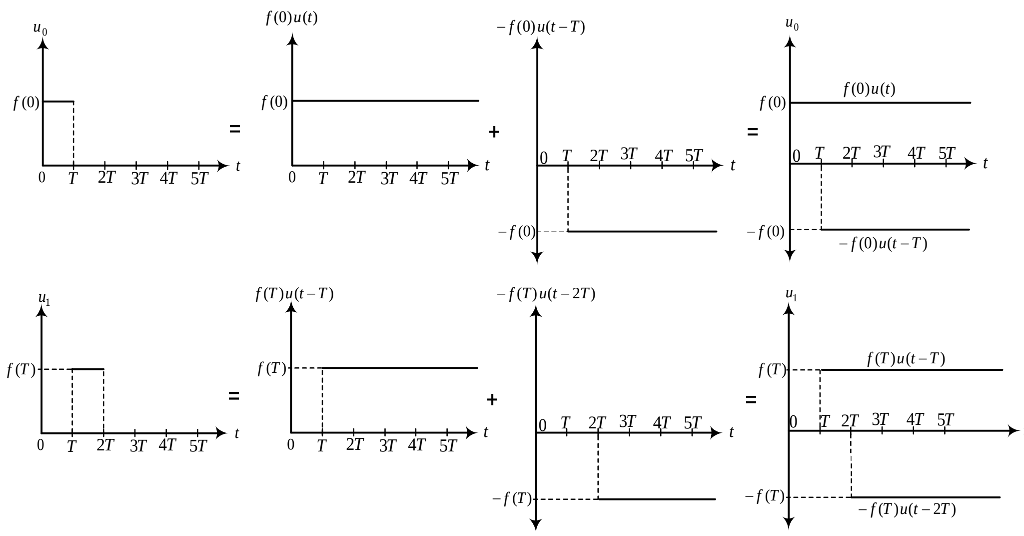


Figure (4) *Decomposition of pulse functions* u_0 *and* u_1

Continuous-time Plant Driven by a Zero-order Hold:

Figure (5) shows a continuous-time plant represented by transfer function $G(s)$, driven by a zero-order hold (the D/A converter) and followed by an output sampler (A/D) converter.

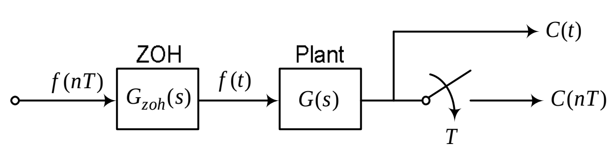


Figure (5) *Continuous-time plant to be digitally controlled.*

For the Fig.(5), the Laplace relationships are

$$C(s) = G(s)F(s)$$
$$F(s) = G_{zoh}(s)F^*(s)$$

Substituting the second equation into first equation will gives

$$C(s) = G_{zoh}(s)\; G(s)\; F^*(s)$$

Starring yields

$$C^*(s) = G_{zoh}G^*(s)\; F^*(s)$$

The corresponding z transform is

$$C(z) = G_{zoh}\; G(z)\; F(z) \qquad (3)$$

where

$$G_{zoh}G(z) = Z\{G_{zoh}(s)\; G(s)\} = Z\left\{\frac{(1-e^{-Ts})}{s}\; G(s)\right\}$$

since $z = e^{Ts}$, then

$$G_{zoh}G(z) = Z\left\{\frac{(1-z^{-1})}{s}\; G(s)\right\}$$

or

$$G_{zoh}G(z) = (1-z^{-1})\; Z\left\{\frac{G(s)}{s}\right\} \qquad (4)$$

Ex1: If $G(s) = \frac{k}{(s+a)}$ in Fig.(5), find the pulse transfer function $\frac{C(z)}{F(z)}$.

From Eq.(3) and (4), the pulse transfer function is

$$\frac{C(z)}{F(z)} = G_{zoh}\; G(z) = (1-z^{-1})\; Z\left\{\frac{G(s)}{s}\right\}$$

making a partial fraction and the application of z transform table yields

$$\frac{C(z)}{F(z)} = (1-z^{-1})\; Z\left\{\frac{K}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)\right\}$$

Ex2: If $G(s) = \frac{1}{s^2}$ in Fig.(5), find the pulse transfer function $\frac{C(z)}{F(z)}$.

Consulting a table of z transform gives

Ex3: Find the pulse transfer function $\frac{C(z)}{R(z)}$ for the sampled-data system shown in Fig.(6).

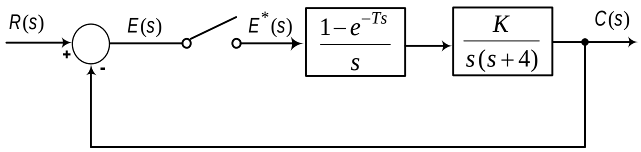


Figure (6) *Sampled-data system with a zero-order hold*

The overall open-loop transfer function when zero hold is included is

To determine $G(z)$ when $G(s)$ contains a $(1-e^{-Ts})$ factor, first we decompose $G(s)$ into $G_1(s)$ and $G_2(s)$ as follows

$$G(s) = G_1(z)G_2(s)$$

where $G_1(s) = (1-e^{-Ts})$ and $G_2(s)$ is the remaining portion of $G(s)$. The function $G_1(s)$ is the Laplace transform of a unit impulse at the origin and a negative unit impulse at t=T. The corresponding $g_1(t)$ is shown in Fig.(7). Because this time function $g_1(t)$ exists only at the sampling instants, the sampled function $g_1^*(t)$, will be the same as $g_1(t)$. Thus,

$$G_1(s) = G_1^*(s)$$

Substitution of this result into equation of $G(s)$ shows that

$$G(s) = G_1^*(s)G_2(s)$$

Starring gives

$$G^*(s) = G_1^*(s)\, G_2^*(s)$$

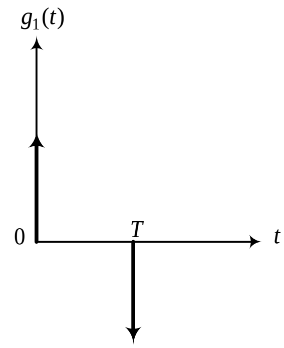


Figure (7) *Time function* $g_1(t) = \mathcal{L}^{-1}[G_1(z)] = \mathcal{L}^{-1}(1 - e^{-Ts})$

The corresponding z transform is

$$G(z) = G_1(z)\, G_2(z) = (1 - z^{-1})\, G_2(z)$$
$$= \frac{(z-1)}{z} G_2(z)$$

Since

Thus

Substitution this result into $G(z)$ equation

$$G(z) = G_1(z)\, G_2(z) = \frac{K}{16}\left(\frac{4T}{z-1} - 1 + \frac{z-1}{z - e^{-4T}}\right)$$

For T=1/4, then $G(z)$ becomes

Numerical Integration

The fundamental concept is to represent the given filter transfer function H(s) as a differential equation and to derive a difference equation whose solution is an approximation of the differential equation. For example, the system

$$\frac{U(s)}{E(s)} = H(s) = \frac{a}{s+a} \qquad (1)$$

is equivalent to the differential equation

$$\dot{u} + a\,u = a\,e$$

Now, if one write Eq.(1) in integral form

$$u(t) = \int_0^t [-a\,u(\tau) + a\,e(\tau)]\,d\tau$$

$$u(kT) = \int_0^{kT-T} [-a\,u(\tau) + a\,e(\tau)]\,d\tau + \int_{kT-T}^{kT} [-a\,u(\tau) + a\,e(\tau)]\,d\tau$$

$$u(kT) = u(kT-T) + \{\ Area\ of\ (-au + a\,e)\ over\ kT - T \le \tau \le kT \qquad (2)$$

Many rules have been developed based on how the incremental area term is approximated. Three possibilities are sketched in Fig.(1).

1. Forward rectangular rule:

In this rule, we approximate the area by the rectangle looking forward from *kT-T* and the amplitude of the rectangle to be the value of the integrand at *kT-T*. The width of the rectangle is T. The result is an equation in the first approximation:

$$u(kT) = u(kT-T) + T\,[-a\,u(kT-T) + a\,e(kT-T)]$$

$$= (1 - aT)\ u(kT-T) + aT\ e(kT-T)]$$

The transfer function corresponding to the forward rectangular rule in this case is

$$H_F(z) = \frac{aT\,z^{-1}}{1-(1-aT)\,z^{-1}} = \frac{a}{(z-1)/T+a} \quad \text{(Forward rectangular rule)}$$

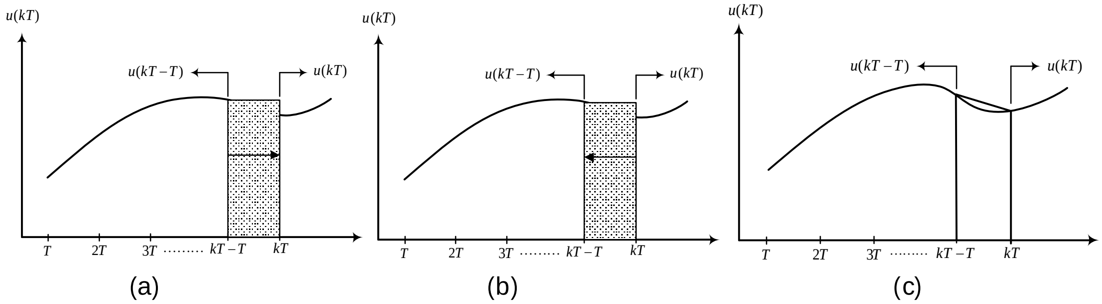


Figure (1) *Sketches of three ways. The area under the curve from kT to kT+1 can be approximated (a) Forward rectangular rule (b) Backward rectangular rule (c) Bilinear or trapezoid rule*

2. Backward rectangular rule:

A second rule follows from taking the amplitude of the approximating rectangle to be the value looking backward from *kT* toward kT-T, namely, *-a u(kT)+ae (kT)*. The equation for *u* is

$$\begin{aligned} u(kT) &= u(kT-T) + T[-a\,u(kT) + a\,e(kT)] \\ &= \frac{u(kT-T)}{1+aT} + \frac{aT}{1+aT} e(kT) \end{aligned}$$

Again we take the z-transform and compute the transfer function of the *backward rectangular rule*

$$\begin{aligned} H_B(z) &= \frac{aT}{1+aT}\ \frac{1}{1-z^{-1}/(1+aT)} = \frac{aT\,z}{z\,(1+aT)-1} \\ &= \frac{a}{(z-1)/Tz+a} \quad \text{(backward rectangular rule)} \end{aligned}$$

3. Trapezoid rule (Bilinear Transformation):

The final version of integration rules is the trapezoid found by taking the area approximated in Eq.(2) to be the trapezoid formed by the average of the previously selected rectangles. The approximating difference equation is

$$u(kT) = u(kT-T) + \frac{T}{2}[-a\,u(kT-T) + a\,e(kT-T) - a\,u(kT) + a\,e(kT)]$$

$$= \frac{1-(aT/2)}{1+(aT/2)}\ u(kT-T) + \frac{aT/2}{1+(aT/2)}\,[\,e(kT-T)+e(kT)\,]$$

The corresponding transfer function from the trapezoid rule is

$$H_T(z) = \frac{aT\,(z+1)}{(2+aT)\,z+aT-2} = \frac{a}{(2/T)\,[(z-1)/(z+1)]+a} \quad \text{(trapezoid rule)}$$

One can tabulate the above obtained results in Table (1). We can see the effect of each of our methods is to present a discrete transfer function that can be obtained from the given Laplace transfer function $H(s)$ by substitution of an approximation for the frequency variable. Each of the approximations given in Table (1) can be viewed as a map from the s-plane to the z-plane.

Table (1)

H(s)	Method	Transfer function	Approximation
$\frac{a}{s+a}$	Forward rule	$H_F = \frac{a}{(z-1)/T+a}$	$s \leftarrow \frac{z-1}{T}$
$\frac{a}{s+a}$	Backward rule	$H_F = \frac{a}{(z-1)/Tz+a}$	$s \leftarrow \frac{z-1}{T\,z}$
$\frac{a}{s+a}$	Trapezoid rule	$H_F = \frac{a}{(2/T)[(z-1)/(z+1)]+a}$	$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$

Since the (s= $j\omega$)-axis is the boundary between poles of stable systems and poles of unstable systems, it would be interesting to know how the $j\omega$-axis is mapped by the three rules and where the left (stable) half of the s-plane appears in the z-plane. For this purpose we must solve the relations in for z in terms of *s*. We find

1. $z = 1 + T\,s$ (Forward rectangular rule)
2. $z = \frac{1}{1-Ts}$ (backward rectangular rule)
3. $z = \frac{1+Ts/2}{1-Ts/2}$ (trapezoid rule)

If we let s= $j\omega$ in these equations, we obtain the boundaries of the regions in the z-plane which originate from the stable portion of the s-plane. The shaded areas sketched in Fig.(2) are these stable regions for each case.

Because the unit circle is the stability boundary in the z-plane, it is apparent from Fig.(2-a) that the forward rectangular rule could cause a stable continuous filter to be mapped into an unstable digital filter.

To see how points map from s plane to the z plane under backward mapping, the z expression for bilinear can be written as ($1/2$ is added to and subtracted from the right-hand side)

$$z = \frac{1}{2} + \left\{ \frac{1}{1-Ts} - \frac{1}{2} \right\} = \frac{1}{2} - \frac{1}{2}\frac{1+Ts}{1-Ts}$$

or

$$\left| z - \frac{1}{2} \right| = \left| -\frac{1}{2}\frac{1+Ts}{1-Ts} \right|$$

Consider a point $s = \sigma + j\omega$ where $-\infty < \omega < \infty$. Then

$$\left| z - \frac{1}{2} \right| = \frac{1}{2}\left| \frac{1+T(\sigma + j\omega)}{1-T(\sigma + j\omega)} \right| = \frac{1}{2}\frac{|(1+T\sigma) + j\omega|}{|(1-T\sigma) + j\omega|} = \frac{1}{2}\frac{\sqrt{(1+T\sigma)^2 + \omega^2}}{\sqrt{(1-T\sigma)^2 + \omega^2}}$$

Now it is easy to see that with $\sigma = 0\,(s = j\omega)$, the magnitude of $z - 1/2$ is constant

$$\left| z - \frac{1}{2} \right| = \frac{1}{2}$$

and the curve of rule (2) is thus a circle as drawn in Fig.(2-b). Also, if $\sigma < 0$, then $|(1+T\sigma) + j\omega| < |(1-T\sigma) + j\omega|$ and $|z - 1/2| < 1/2$. On the other hand if $\sigma > 0$, then $|(1+T\sigma) + j\omega| > |(1-T\sigma) + j\omega|$ and $|z - 1/2| > 1/2$. Therefore, it is clear that the backward rule maps the stable region of the s-plane into a circle of radius 0.5 inside the unit circle (stable region) of the z-plane, as shown in Fig.(2-b).

To see how points map from s plane to the z plane under bilinear mapping, the z expression for bilinear can be written as

$$z = -\frac{s + (2/T)}{s - (2/T)}$$

Then

$$z = -\frac{\sigma + j\omega + (2/T)}{\sigma + j\omega - (2/T)} = -\frac{(\sigma + 2/T) + j\omega}{(\sigma - 2/T) + j\omega}$$

and

$$|z| = \left| -\frac{(\sigma + 2/T) + j\omega}{(\sigma - 2/T) + j\omega} \right| = \frac{|(\sigma + 2/T) + j\omega|}{|(\sigma - 2/T) + j\omega|}$$

Note for $\sigma < 0$, then $|(\sigma + 2/T) + j\omega| < |(\sigma - 2/T) + j\omega|$ and $|z| < 1$. On the other hand if $\sigma > 0$, then $|(\sigma - 2/T) + j\omega| < |(\sigma - 2/T) + j\omega|$ and $|z| > 1$. Also, if $\sigma = 0$ then

$$|z| = \frac{|(2/T) + j\omega|}{|(2/T) + j\omega|} = 1$$

Therefore, it is interesting to notice that the bilinear rule maps the stable region of the s-plane exactly into the stable region of the z-plane (see. Fig.(2-c)).

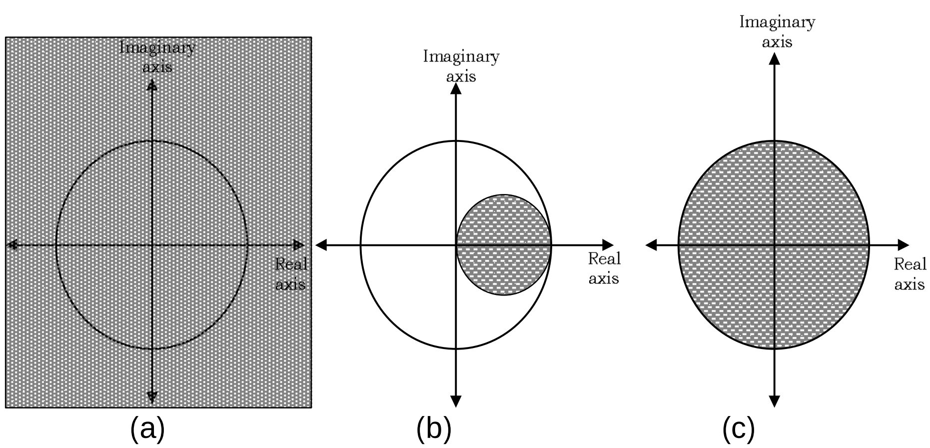


Figure (2) *Maps of the left-half of the s-plane by the integration rules into the z-plane. Stable s-plane poles map into the shaded regions in the z-plane. (a) Forward rectangular rule (b) Backward rectangular rule (c) Bilinear or trapezoid rule*

The original $H(s)$ had a pole at $s = -a$, and for real frequencies, $s = j\omega$, the magnitude of $H(j\omega)$ is given by

$$\left|H(j\omega)\right|^2 = \frac{a^2}{\omega^2 + a^2} = \frac{1}{\omega^2 / a^2 + 1}$$

Thus our reference filter has a half-power point, $|H|^2 = 1/2$, at $\omega = a$. It will be interesting to know where $H_T(z)$ a half-power point has.

Signals with poles on the imaginary axis in the s-plane (sinusoids) map into signals on the unit circle of the z-plane. A sinusoid of frequency ω_1 corresponds to $z_1 = e^{j\omega_1 T}$, and the response of $H_T(z)$ to a sinusoid of frequency ω_1 is $H_T(z_1)$.

$$H_T(z_1) = \frac{a}{\left(\frac{2}{T}\,\frac{e^{j\omega_1 T} - 1}{e^{j\omega_1 T} + 1} + a\right)} = \frac{a}{\left(\frac{2}{T}\,\frac{e^{j\omega_1 T/2} - e^{-j\omega_1 T/2}}{e^{j\omega_1 T/2} + e^{-j\omega_1 T/2}} + a\right)}$$

$$= \frac{a}{\left(\frac{2}{T}\; j\tan\left(\frac{\omega_1 T}{2}\right) + a\right)}$$

The magnitude squared of H_T will be $\frac{1}{2}$ when

$$\frac{2}{T}\tan\left(\frac{\omega_1 T}{2}\right) = a$$

or

$$\tan\left(\frac{\omega_1 T}{2}\right) = \frac{aT}{2} \qquad (3)$$

The latter equation is a measure of the frequency distortion or warping caused by Tustin's rule. Whereas we wanted to have a half-power at $\omega = a$, we realized a half-power at $\omega_1 = (2/T)\tan^{-1}(aT/2)$. ω_1 will be approximately correct only if $aT/2 << 1$ so that $\tan^{-1}(aT/2) \cong aT/2$, that is, if $\omega_s (= 2\pi/T) >> a$ and the sample rate is much faster than the half-power frequency.

We can turn our attentions around and suppose that we really want the half-power point to be at ω_1. Equation can be made into an equation of prewarping: If we select a according to Eq.(3), then, using bilinear rule for the design, the half-power point will be at ω_1. A statement of a complete set of rules for filter design via bilinear transformation with prewarping is

1. Write the desired filter characteristic with transform variable s and critical frequency ω_1 in the form $H(s/\omega_1)$.
2. Replace ω_1 by a such that

$$a = \frac{2}{T}\tan\left(\frac{\omega_1 T}{2}\right)$$

and in place of $H(s/\omega_1)$, consider the prewarped function $H(s/a)$. For more complicated shapes, such as bandpass filters, the specification frequencies, such as band edges and center frequency, should be prewarped before the continuous design is done; and then the bilinear transformation will bring all these points to their correct frequencies in the digital filter.

3. Substitute

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

in $H(s/a)$ to obtain the prewarped equivalent $H_p(z)$. As a frequency substitution the result can be expressed as

$$H_p(z) = H\left(\frac{s}{\omega_1}\right)\Bigg|_{\frac{s}{\omega_1} = \frac{1}{\tan(\omega_1 T/2)}\frac{z-1}{z+1}} \qquad (4)$$

It is clear from Eq.(4) that when $\omega = \omega_1$, $H_p(z_1) = H(j1)$ and the discrete filter has exactly the same transmission at ω_1 as the continuous filter has at this frequency. This is the consequence of prewarping.

Ex1: Apply the method of bilinear transformation to following filter (T=1 sec.):

$$H(s) = \frac{1}{s^2 + 0.2s + 1}$$

We make the following substitution for s in the original s-domain transfer function

$$s = \frac{2}{T}\frac{z-1}{z+1} = 2\frac{(z-1)}{(z+1)}$$

After cleaning up the numerical details, the resulting discete-time transfer function is

$$H(z) = 0.185\left[\frac{(z+1)^2}{z^2 - 1.111z + 0.852}\right]$$

Ex2: Let us first prewarp the poles of the filter considered in the above example. The critical frequency of the filter is $\omega_1 = 1 rad/\sec$.

1. Write the desired filter characteristic with transform variable s and critical frequency ω_1 in the form $H(s/\omega_1)$.

$$H(s) = \frac{1}{(s/\omega_1)^2 + 0.2(s/\omega_1) + 1}$$

2. Replace ω_1 by a such that

$$a = \frac{2}{T}\tan\left(\frac{\omega_1 T}{2}\right) = \frac{2}{1}\tan\left(\frac{1\times 1}{2}\right) = 1.092$$

Then H(s) becomes

$$H(s) = \frac{1}{(s/a)^2 + 0.2(s/a) + 1}$$
$$= \frac{1.1924}{s^2 + 0.218s + 1.1924}$$

3. Substitute $s = \frac{2}{T}\frac{z-1}{z+1}$

Cleaning up the numerical details,

$$H(z) = 0.226\left[\frac{(z+1)^2}{z^2 - 0.997z + 0.845}\right]$$

Hw: The transfer function of a third order low-pass filter designed to have unity pass bandwidth ($\omega_1 = 1 rad/\sec.$) is

$$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

Compute the discrete equivalents and plot the frequency response using forward, backward, and bilinear rules. Use T=0.1, 1 and 2 sec.

Response between sampling instants

Two different time functions which have the same sampled values are illustrated in Fig.(1). The inverse z transform $Z^{-1}[F(z)] = f^*(t)$ yields the value of the function at the sampling instants. The behavior between sampling instants may be determined by synthetic sampler method.

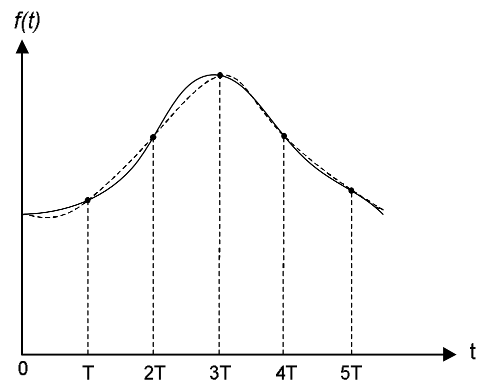


Figure (1) Two functions with the same values at the sampling instants

The dotted box in Fig.(2) represents a fictitious, or synthetic, sampler which is inserted in series with the actual sampler. The sampling rate of the fictitious sampler is m times that of the actual sampler (m=2,3,...). The corresponding period is T/m. At submultiple of the sampling period T/m, when the fictitious sampler is closed, the actual sampler is open. Thus, the fictitious sampler does not affect the operation of the system. The fictitious sampler does not actually exist, but is merely employed as an aid for understanding the following analysis.

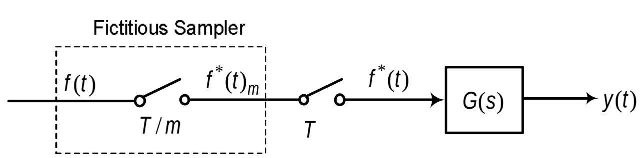


Figure (2) Fictitious sampler

The continuous function can be written as

$$y(t) = \sum_{n=0}^{\infty} f(nT)\,g(t-nT)$$

Submultiple of the sampling period are represented by the term kT/m where $k = 0,1,2,\ldots$. For example, if $m = 3$, the successive intervals kT/m are $0, T/3, 2T/3, 4T/3, \ldots$. The value of the output at the submultiple intervals is obtained by letting $t = kT/m$ in the preceding equation.

$$y\left(\frac{kT}{m}\right) = \sum_{n=0}^{\infty} f(nT)\; g\left(\frac{kT}{m} - nT\right) \tag{1}$$

The output at the submultiples sampling instants $y(kT/m) = y^*(t)_m$ may also be expressed as an impulse train. That is,

$$y^*(t)_m = \sum_{k=0}^{\infty} y\left(\frac{kT}{m}\right) u_1\left(t - \frac{kT}{m}\right)$$

The Laplace transform is

$$Y^*(s)_m = \sum_{k=0}^{\infty} y\left(\frac{kT}{m}\right) e^{-kTs/m}$$

The corresponding z transform is

$$Y(z)_m = \sum_{k=0}^{\infty} y\left(\frac{kT}{m}\right) z^{-k/m}$$

Substituting $y(kT/m)$ from Eq.(1) into the preceding expression gives

$$Y(z)_m = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f(nT)\; g\left[\left(\frac{k}{m} - n\right)T\right] z^{-k/m}$$

$$Y(z)_m = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f(nT)\; z^{-n} g\left[\left(\frac{k}{m} - n\right)T\right] z^{-(k/m-n)} \tag{2}$$

Consider the expansion of a typical term in which $m = 2$ and $n = 3$,

$$\sum_{k=0}^{\infty} f(3T)\; z^{-3}\; g\left[\left(\frac{k}{2} - 3\right)T\right] z^{-(k/2-3)}$$

$$= f(3T)\; z^{-3}\left[g(0)\; z^0 + g\left(\frac{T}{2}\right) z^{-1/2} + g(T)\; z^{-1} + g\left(\frac{3T}{2}\right) z^{-3/2} + \cdots\right]$$

$$= f(3T)\; z^{-3} \sum_{\ell=0}^{\infty} g\left(\frac{\ell T}{m}\right) z^{-\ell/m}$$

For a physically realizable system the impulse response $g(t)$ is zero for negative time. Thus, the first term to appear in the bracket is for k=6, in which case $g[(k/2-3)T]$=g(0). Fom the preceding result, the general form of Eq.(1) is

$$Y(z)_m = \sum_{n=0}^{\infty} f(nT) z^{-n} \sum_{\ell=0}^{\infty} g\left(\frac{\ell T}{m}\right) z^{-\ell/m}$$

$$= F(z) G(z)_m \tag{3}$$

where

$$G(z)_m = [G(z)]_{z=z^{1/m},T=T/m} \tag{4}$$

Thus, $G(z)_m$ is obtained by substituting $z^{1/m}$ for z and T/m for T in $G(z)$. The result given in Eq.(3) may also be obtained by letting $k/m - n = \ell/m$ where $\ell = 0,1,2,\ldots,$ in Eq.(2).

The equation for $G(z)_m$ for the system of Fig.(3) is obtained as follows:

$$C(z)_m = E(z)G(z)_m$$

$$E(s) = R(s) - E^*(s)G(s)H(s)$$

Starring gives

$$E^*(s) = R^*(s) - E^*(s)GH^*(s)$$

Thus,

$$E(z) = R(z) - E(z)\ GH(z)$$

Solving this last equation for $E(z)$ and then substituting $E(z)$ into the first equation gives the desired result.

$$C(z) = \frac{G(z)_m}{1+GH(z)}R(z) \tag{5}$$

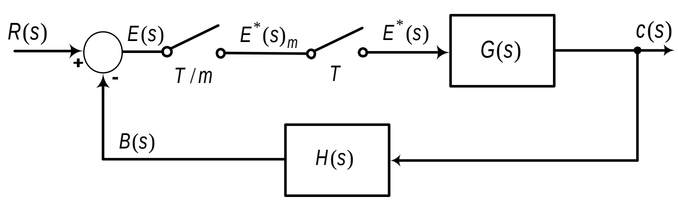


Figure (3) Sampled-data system with a fictitious sampler

<u>EX:</u> Consider that the sample-data system of Fig.(1) has the transfer function

$$G(s) = \frac{1}{s+1}$$

The input $r(t)$ is a unit-step function. The sampling period is 1 second. It is desired to find the response of the system at time instants of $t = kT/3$, $k = 0,1,2,\ldots.$

From Eq.(3), the z-transform of the system at the submultiple-sampling instants is written as

$$Y(z) = F(z)G(z)_m$$

where $m = 3$, and

$$G(z)_m = G(z)\Big|_{z=z^{1/3}, T=T/3} = \frac{z}{z-e^{-T}}\Big|_{z=z^{1/3}, T=T/3}$$

Thus,

The z-transform of the unit-step input is $F(z) = z/(z-1)$. The z-transform of the submultiple-sampled output is

(6)

However, one difficulty remains in that the last expression has fractional powers as well as integral powers of z. To overcome this difficulty, we introduce a new variable w, such that

$$w = z^{1/3}$$

Eq.(6) becomes

Expanding the $C(z)_m$ into a power series in w, we have

The coefficients of the power-series expansion of $y(z)_m$ are the values of $y^*(t)_m$ at $t = kT/3,\ k = 0,1,2,\ldots$. The response $y^*(t)_3$ is shown in Fig.(4). In this case, the value of the submultiple sampling method is clearly demonstrated, since the ordinary z-transform obviously would produce a misleading result.

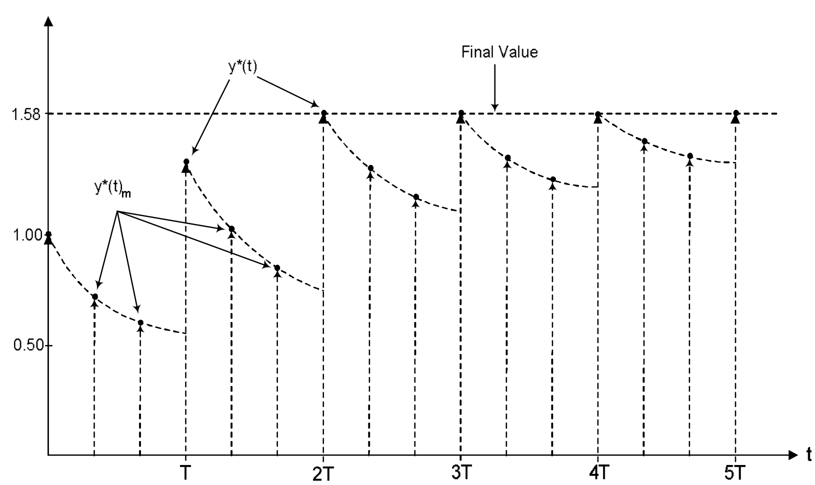


Figure (4) Output responses at t=kT/3

EX: Determine the response midway between the sampling instants for system of Fig.(3 to a unit step-function input and for $G(s)=\dfrac{K}{s(s+4)}$ and $H(s)=1$.

For this system, $GH(z)=G(z)$ is given by

$$G(z)=\frac{z(1-e^{-4T})}{(z-1)(z-e^{-4T})}$$

Replacing T by T/m and z by $z^{1/m}=z^{1/2}$

For K=1 and $T=\dfrac{1}{4}$, this becomes

For K=1 and $T=\dfrac{1}{4}$, the function $G(z)$ is given by

Substitution of these results into Eq.(5), one can obtain

$$C(z)_m=\frac{0.098z^{1/2}}{(z^{1/2}-1)(z^{1/2}-0.607)}\frac{(z-1)(z-0.368)}{[(z-1)(z-0.368)+0.158z]}R(z)$$

To eliminate fractional powers of z, let $w=z^{1/2}$. Thus,

Cross-multiplying yields

(7)

The significance of replacing $z^{1/2}$ by w is seen by noting that because

Then

$$C(w)_m=c(0)+c\left(\frac{T}{2}\right)w^{-1}+c(T)w^{-2}+c\left(\frac{3T}{2}\right)w^{-3}+c(2T)w^{-4}+\cdots$$

Thus, the w sampling instants are the desired submultiple sampling instants. The difference equation associated with Eq.(7) is

$$+0.098r(k-1)-0.134r(k-3)+0.036r(k-5) \quad (8)$$

Because

$$R(z)=r(0)+r(T)z^{-1}+r(2T)z^{-2}+\cdots$$

Then replacing z^{-1} by w^{-2} gives

Thus, r=0 at 1,3,5,…sampling instants of w. Application of Eq.(8) to obtain the values at the submultiple sampling instants gives

c(0)=0

c(1)=

c(2)=

c(3)=

c(4)=

c(5)=

……………………………………………………………………..

Replacing k by kT/2 shows that the response at the sampling instants c(0), c(1), c(3), etc., corresponds to the response at time c(0), c(0.5T), c(T), c(1.5T), etc.

HW: Find the inverse using the long division method. For $R(z)=z/(z-1)$, then $R(w)=w^2/(w^2-1)$. Thus, substituting this value of $R(w)$ into Eq.(8), and then dividing the numerator of $C(w)_m$ by the denominator yields the desired values as the coefficients of the answer.

Time Response

In this section the time response of the sampled data system of Fig.(1) to unit step input will be determined. Three methods will be explained: long-division, difference equations and partial fraction expansion.

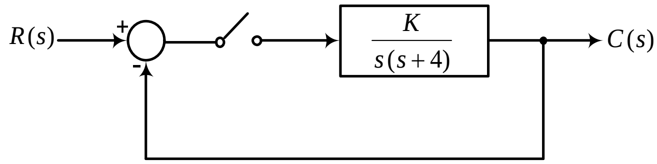


Figure (1) *Sampled data system*

The corresponding z transform of $G(s)$

$$G(z)=\frac{z(K/4)\,(1-e^{-4T})}{(z-1)(z-e^{-4T})}$$

Letting $K=1$ and $T=0.25$ sec, then

$$G(z)=\frac{0.158\,z}{(z-1)(z-0.368)}$$

The pulse transfer function $\frac{C(z)}{R(z)}$ is

$$\frac{C(z)}{R(z)}=\frac{G(z)}{1+G(z)}$$

$$C(z)=\frac{0.158\,z}{[(z-1)(z-0.368)+0.158z]}R(z)=\frac{0.158\,z}{(z-0.61)^2}R(z)$$

- Long division method:

For unit step input, $R(z)=\frac{z}{z-1}$. Then

$$C(z)=\frac{0.158\,z^2}{(z-0.61)^2(z-1)}$$

Using the long-division method to determine the inverse gives

$$\begin{array}{r|l} & 0.158\,z^{-1}+0.349\,z^{-2}+0.522\,z^{-3}+\cdots \\ z^3-2.21z^2+1.58z-0.368 & 0.158z^2 \end{array}$$

Because

then

c(0)=0,   c(T)=0.158,   c(2T)=0.349,   and   c(3T)=0.522

A plot of the response $c(nT)$ at the sampling instants is shown in Fig.(2).The long division method becomes quite cumbersome for computing $c(nT)$ for larger values of *n*. A more convenient procedure results from expressing the solution in the form of a difference equation.

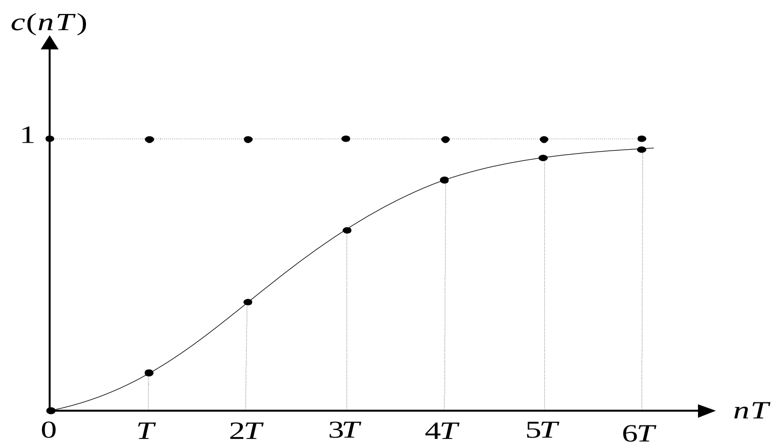


Figure (2) *Sampled data system*

Difference Equations:

To determine the inverse z transform by this method, one can write the equation for $C(z)$ in the form

$$C(z) = \frac{0.158\,z}{[z^2 - 1.21z + 0.368]}\ R(z)$$

Thus

$$C(z) - 1.21z^{-1}C(z) + 0.368z^{-2}C(z) = 0.158z^{-1}R(z)$$

Application of right shifting property

$$Z[\,f(nT - kT) = z^{-n}F(z)$$

Then the preceding expression yields directly the difference equation

$$c(nT) = 1.21\,c(nT - T) - 0.368\,c(nT - 2T) + 0.158\,r(nT - T)$$

This difference equation gives the value $c(nT)$ at the nth sampling instants in terms of values at the preceding sampling instants. Application of this result to obtain the values at the sampling instants gives

$$c(0) = 0,$$
$$c(T) = 0.158\ r(0) = 0.158$$
$$c(2T) = 1.21 c(T) + 0.158\ r(T) = 0.349$$
$$c(3T) = 1.21\ c(2T) - 0.368\ c(T) + 0.158\ r(2T) = 0.522$$

Such recurrence relationships lend themselves very well to solution by a digital computer.

- Partial-fraction expansion:

The response $c(nT)$ at the sampling instants may be also be obtained by performing a partial fraction expansion and then inverting. Thus

$$C(z) = z\left[\frac{0.158z}{(z-1)(z-0.61)^2}\right] = z\left[\frac{A}{(z-1)} + \frac{B_1}{(z-0.61)^2} + \frac{B_2}{(z-0.61)}\right]$$

The partial-fraction expansion constants are $A = 1$, $B_1 = -0.24$, and $B_2 = -1.0$. Thus, $C(z)$ becomes

$$C(z) = \frac{z}{(z-1)} - 0.39\frac{0.61z}{(z-0.61)^2} - \frac{z}{(z-0.61)}$$

By noting that

$$Z^{-1}\left[\frac{z}{z-1}\right] = 1, \qquad Z^{-1}\left[\frac{z}{z-a}\right] = a^{nT}, \quad \text{and} \qquad Z^{-1}\left[\frac{az}{(z-a)}\right] = nT\ a^{nT}$$

The inverse is found to be

With this method, the value $c(nT)$ at any sampling instants may be calculated directly without the need to compute the value at all the preceding instants.

Mapping of s-plane to z-plane

It is possible to map fro the s plane to the z plane using the relationship

$$z = e^{sT}$$

Now

$$s = a \pm jb$$

Therefore,

$$z = e^{sT} = e^{(a \pm jb)T} = e^{aT}\, e^{\pm jbT} = r\, e^{j\theta} \qquad (1)$$

where $r = |z| = e^{aT}$ and $\theta = bT$.

Since $\omega_s = 2\pi f_s = \frac{2\pi}{T}$, then $\theta = bT = \left(\frac{2\pi}{T}\right) T = 2\pi$, where ω_s is the switching frequency in rad/sec and f_s is the switching frequency in Hz. Equation (1) results in a polar diagram in the z plane as shown in Fig.(1).

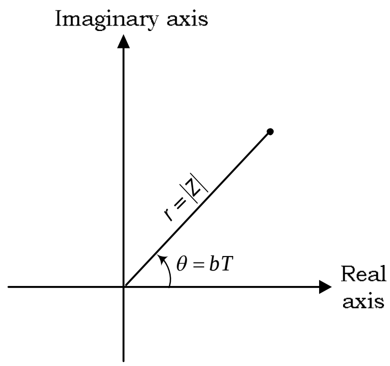


Figure (1) *Mapping from the s to the z plane*

Two horizontal lines of constant of constant b are shown in the s-plane of Fig.(2.a). The corresponding paths in the z-plane are radial straight lines.

$$z = e^{sT} = e^{(a \pm jb)T} = e^{aT}\, e^{\pm jbT}$$

The angle of inclination of these radial lines is $\theta = \pm bT$.

Two vertical lines of constant a (i.e. constant settling time) are shown in Fig.(2.b). The corresponding paths in the z-plane are circles of radius e^{aT}. For negative values of a the circles are inside the unit circle of the z-plane. For positive values of a the circles lie outside the unit circle of the z-plane. Thus, one can conclude that the left-hand side (stable) of the s plane corresponds to a region within a circle of unity radius (the unit circle) in the z plane.

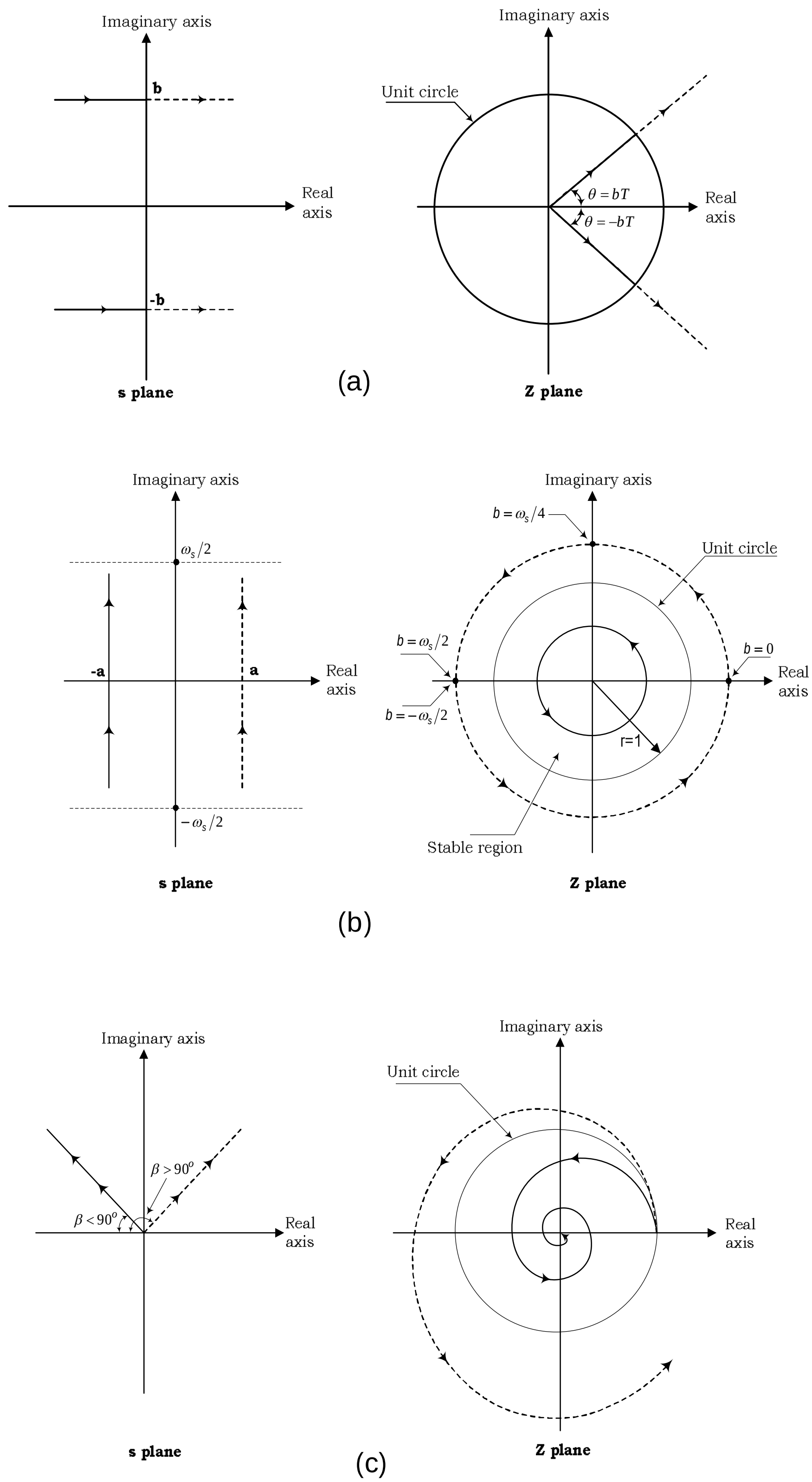


Figure (2) *Corresponding paths in the s plane and z plane*

Radial lines of constant damping ratio $\zeta = \cos\beta$ are shown in Fig.(1.c). In polar coordinates, $s = a \pm jb = -\zeta\,\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2}$. Thus,

$$z = e^{-\zeta\omega_n T} e^{\pm j\omega_n T\sqrt{1-\zeta^2}}$$

The corresponding paths in the z-plane are logarithmic spirals. For $\beta < 90^0$ the spirals decay within the unit circle, and for $\beta > 90^0$ the spirals grow outside the unit circle.

Consider now how a given point, $z = r\,e^{j\theta}$, in the z plane maps back into the s plane. For

$$z = r\,e^{j\theta} = e^{sT} = e^{(a\pm jb)T}$$

Equating real and imaginary parts shows that

$$\ln(r) = aT$$

$$\theta = \pm bT$$

This verifies the fact that a circle of constant *r* in the z plane is a vertical line of constant a in the s plane. Similarly, a ray at angle θ in the z plane is a horizontal line of constant b in the s plane.

<u>Ex</u>: Find the corresponding locations of points in the s-plane into z plane

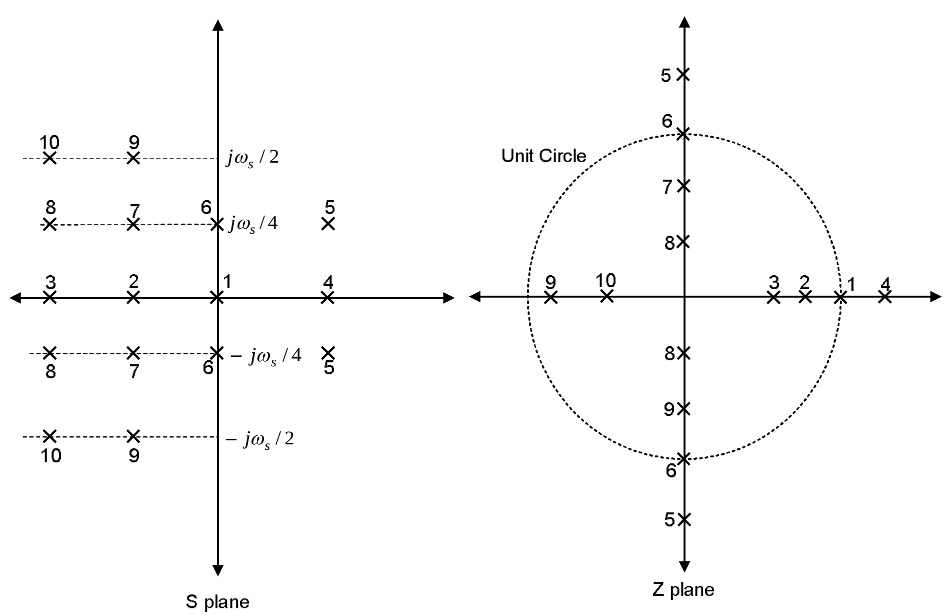


Figure (3) *Corresponding pole locations between the s plane and the z plane*

Since $\Rightarrow r = e^{aT}$ and

Points 3, 2, 1 and 4:

$$b_1, b_2, b_3 \ and \ b_4 = 0 \Rightarrow \quad \theta = bT \quad \Rightarrow \quad \theta_{1,2,3,4} = 0$$

$$a_1 = 0 \Rightarrow \ r_1 = e^{0T} = 1,$$

and $\Rightarrow r_2\,(e^{a_2 T}) > r_3(e^{a_3 T}),$

$$a_4 > 0 \Rightarrow$$

Upper points 10 and 9:

$$b_9 \ and \ b_{10} = \frac{\omega_s}{2} = \frac{2\pi/T}{2} = \frac{\pi}{T} \ \Rightarrow \ \theta_9 = b_9 T = \pi \ \text{ and } \ \theta_{10} = b_{10} T = \pi$$

and $a_9 > a_{10} \Rightarrow$

Upper points 10 and 9:

$$b_9 \ and \ b_{10} = -\frac{\omega_s}{2} = -\frac{2\pi/T}{2} = -\frac{\pi}{T} \Rightarrow \ \theta_9 = b_9 T = -\pi \ \text{ and } \ \theta_{10} = b_{10} T = -\pi$$

and $a_{10} < a_9 \Rightarrow$

Therefore, the lower and upper points 9 and 10 coincide on each other. The same argument may be performed with the other points. This results in the corresponding points at z plane.

Ex: The time-response characteristics of the z-plane pole locations are illustrated in Fig.(4). Since $z = e^{sT}$, the response characteristics are a function of both s and T.

The poles in the s-plane occur at $s = a \pm jb$. These poles result in a system transient-response term of the form $k_1\, e^{at} \cos(\omega t \pm \phi)$. When sampling occurs, these s-plane poles result in z-plane poles at

$$z = e^{sT}\Big|_{s = a \pm jb} = e^{aT}\, e^{\pm jbT} = r\, e^{\pm j\theta}$$

The roots of the characteristic equation that appear at $z = r\,e^{\pm j\theta}$ result in a transient response term of the form

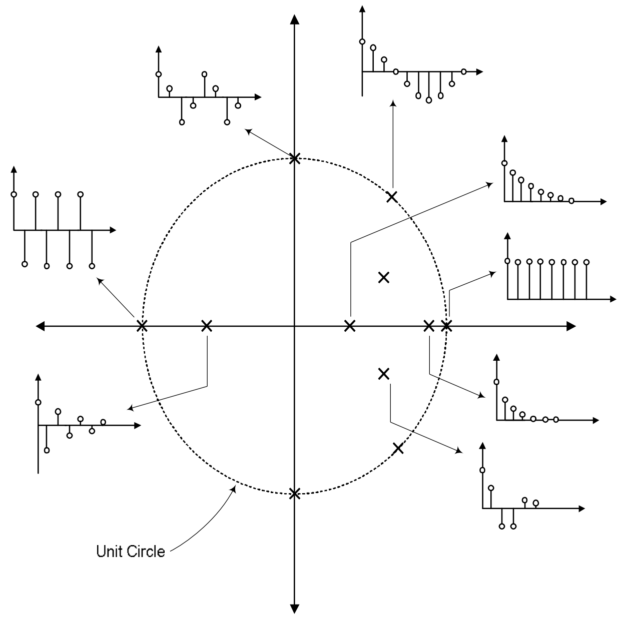


Figure (4) *Transient response characteristics of the z plane pole locations.*

<u>Lines of Constant Damping Ratio</u> ζ :

In the s-domain, the lines of constants damping ratio ζ are rays originating at the origin while the curves representing constant undamped natural frequency ω_n are quarter circles, as shown in Fig.(5).

Figure (6) shows the real and imaginary parts of the complex variables s expressed in terms of ζ and ω_n. That is

$$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

The equivalent point in the z-plane is found by applying the transformation $z = e^{sT}$ to obtain

$$z = e^{-\zeta\omega_n T} e^{j\omega_n T\sqrt{1-\zeta^2}} \qquad (2)$$

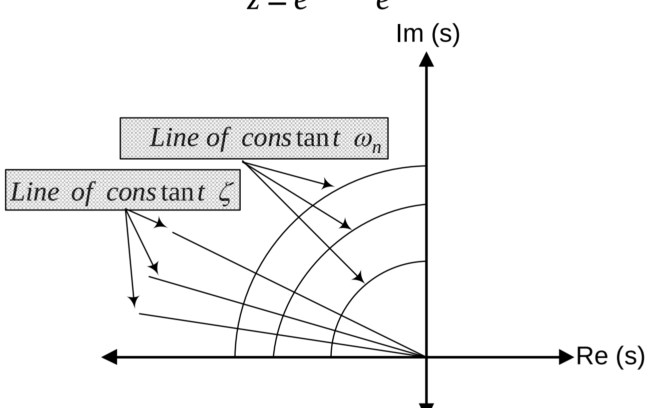


Figure (5) *Lines of constant* ω_n *and curves of constant* ζ

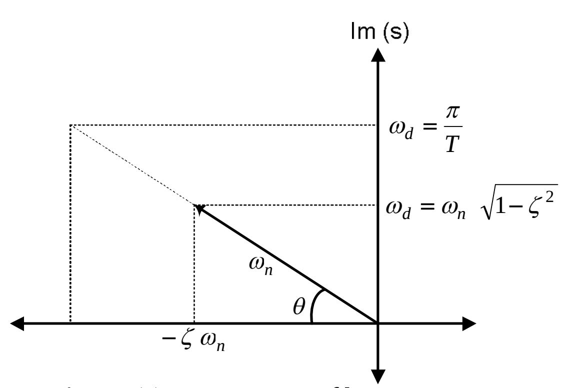


Figure (6) *Components of line of constant* ω_n

If in Eq.(2) we fix ζ and vary ω_n we will plot a log spiral curve, since the magnitude of z will vary exponentially with ω_n, while the phase varies linearly. As shown in Fig.(6), we only need consider the portion of the ray of constant damping ratio between the origin and the point where the ray intersects the edge of the primary strip.

For $\omega_n = 0$ a ray of constant damping ratio starts at the point

$$z = e^0 = 1$$

The other end of the array in the s plane touches the edge of the primary strip. At the point of intersection

$$\omega_n \sqrt{1-\zeta^2} = \frac{\pi}{T}$$

or, equivalently,

$$\omega_n = \frac{\pi}{T\sqrt{1-\zeta^2}}$$

Hence

Thus z is a vector of length

and angle 180^0. Note that the larger ζ the shorter the length of the vector.

The log spiral curves connecting the end points of the curves for $\zeta = 0.1, 0.2, \ldots, 0.9$ in increments of 0.1 are shown in Fig.(7).

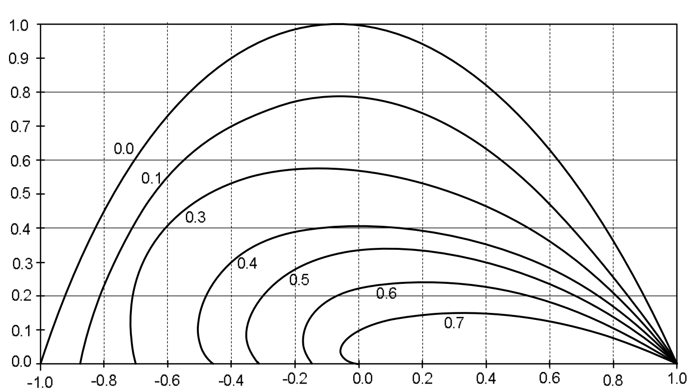


Figure (7) *curves of constant* ζ

<u>Curves of constant natural frequency</u> ω_n:

To find the curves of constant ω_n we again use the transformation $z = e^{sT}$, but this time we fix ω_n and vary ζ. It is customary to let

$$\omega_n = \frac{k\pi}{10T} \qquad\qquad k = 0, 1, 2, \ldots, 10$$

Then

(3)

Eq.(3) can be used to plot the curves of constant ω_n by holding ω_n constant and varying ζ between zero and one. When $\zeta = 0$, corresponding to $s = jk\pi/10$

$$z = e^0 \, e^{j\,k\,\pi\sqrt{1-0^2}/10}$$

In this case z, is a vector of length one and angle $k\pi/10$ rad. Thus, all the curves of constant ω_n originate on the unit circle at the angles

$$k \times \frac{180^0}{10} = k \times 18^0, \qquad\qquad k = 1,\ 2,\ 3,\ \ldots,\ 10.$$

At the other end of each these curves, $\zeta = 1$, and

$$z = e^{-\,k\,\pi\;T/10T} \quad e^{j\,k\,\pi\,T\sqrt{1-1}/10T} = e^{-\,k\,\pi/10} \qquad\qquad k = 1,\ 2,\ 3,\ \ldots,\ 10.$$

These points lie on the positive real axis in the z plane. The smaller k, the larger $e^{-k\pi/10}$. The curves that connect these end points are shown in Fig.(8).

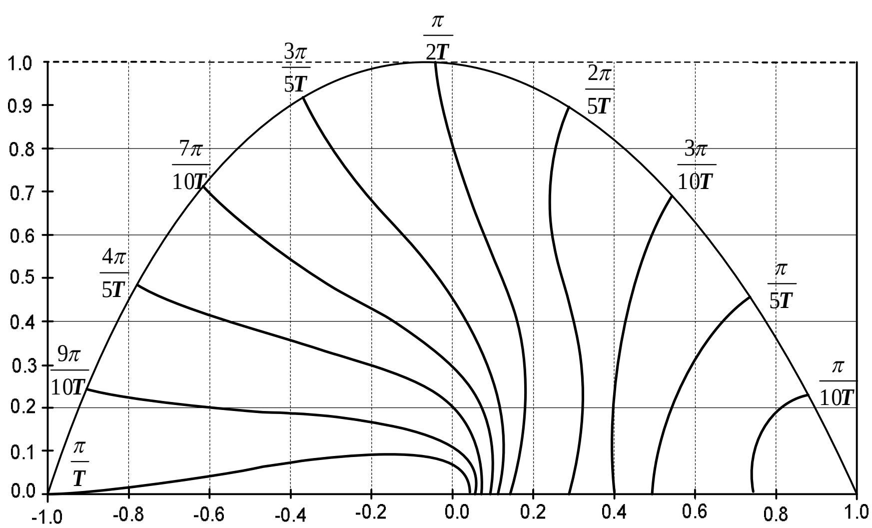


Figure (8) *curves of constant* ω_n

We see that the curves show increasing distortion as k increases. For k=1, the curve is very close to a quarter circle centered at z=1. The curves for k=2 and k=3 still have the general shape of a quarter circle, but for k>3 they do not.

<u>Ex</u>: Map the shaded area in Fig.(9) in the s-plane into corresponding poles in the z plane. In Fig.(8), the lines of constant ω_n are labeled

$$\frac{\pi}{T}, \frac{\pi}{5T}, \frac{3\pi}{10T}, \ldots, \frac{\pi}{T}$$

indicating the value of ω_n that corresponds to each curve. As noted, the curves end at the angles

By combining the curves of constant ζ and constant ω_n we can locate points in the z plane with any desired combination of damping ratio and natural frequency.

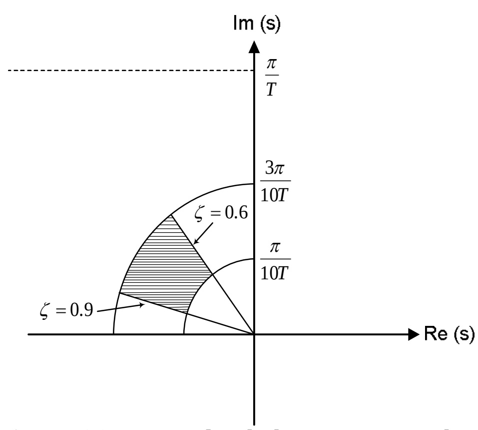


Figure (9) *Desired pole locations in s-plane*

In Fig.(9), poles locations in the s plane with damping rations between 0.6 and 0.9 and natural frequencies between $\pi/10T$ and $3\pi/10T$ are in the shaded area. The corresponding poles in the z plane are shown in Fig.(10).

Notes:

- The curves of constant ζ do not depend on T

- The curves of constant ω_n depends on T, and then, on the sampling rate. For instance, for a sampling rate of 10 Hz, the poles in the shaded region will have natural frequencies between 0.5 and 1.5 Hz, or one tenth and three tenths of the maximum frequency that can be sampled without aliasing, namely, 5 Hz. On the other hand, if the sampling rate is 100 Hz poles in this same region will have natural frequencies between 5 and 15 Hz.

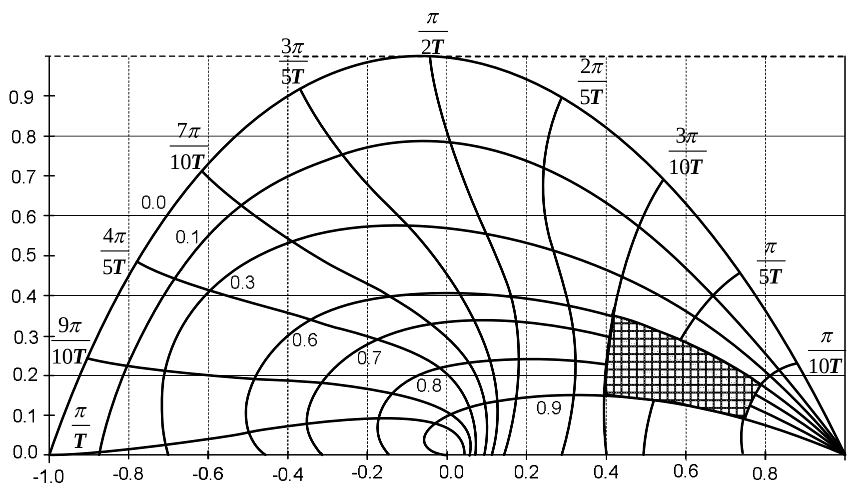


Figure (10) *Desired pole locations in z-plane*

<u>The primary strip</u>:

Suppose we map the primary strip of the s plane into the z plane. We begin by mapping the points of a vertical line

$$s = a + jb$$

where $a < 0$ is fixed. Under the mapping $z = e^{sT}$, a point on this line maps to

$$z = e^{(a+jb)T} = e^{aT}\ e^{jbT}$$

The term is e^{aT} is a real number that can be thought of as a scaling factor for the unit phasor e^{jbT}. If $-\pi/T \le b \le \pi/T$, and a is fixed, with $a < 0$, then the mapping of this portion of the vertical line in the s plane to the z plane is a circle with radius $e^{-aT} < 1$ as shown in Fig.(11). If $a > 0$, the line segment maps to a circle with radius greater than one, as shown in the figure. It should be noted that

$$-\pi/T \le b \le \pi/T \ \Rightarrow\ -\pi \le \theta \le \pi$$

The area confined between $-\pi/T \leq b \leq \pi/T$ is called ***the primary strip***. One can easily see from Fig.(11) that the width of the primary strip is $2\pi/T$. The other strips of the same width as that of primary strip are called the secondary strips.

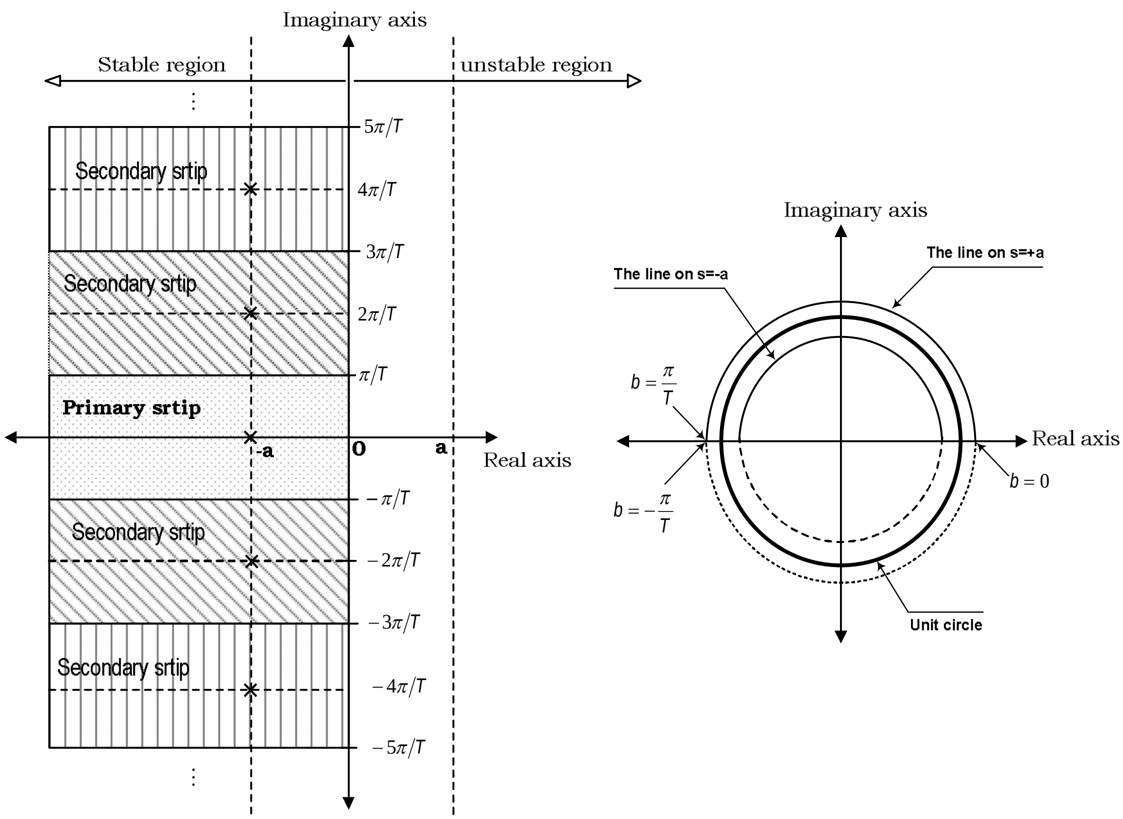


Figure (11) *Mapping of the Primary strip into z-plane*

Routh-Hurwitz Criterion to Discrete-Data System

Since the stability boundary in the z-plane is unit circle $|z|=1$, then to apply this criterion, it is necessary to transform the unit circle of the z-plane to the vertical imaginary axis of the λ plane. This is accomplished by the transformation

$$\lambda = \frac{z+1}{z-1}$$

Solving for z gives

$$z = \frac{\lambda+1}{\lambda-1} \qquad (1)$$

This will transform the interior of the unit circle onto the left half of the λ-plane. When the characteristic equation is expressed in terms of λ, then the Routh-Hurwitz criterion can be applied in the same manner as for the continuous system.

Ex1: For a sampling time period $T = 1/4$ s, determine the value of K such that the system shown in Fig.(1) becomes unstable. That is, roots of the characteristic equation lie on the unit circle of the z –plane (i.e., the imaginary axis of the λ plane.

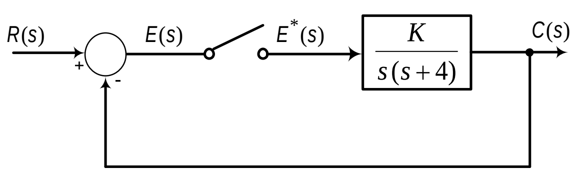


Figure (1) *Sampled-data system*

For T=1/4, the corresponding z-transform of $G(s) = \dfrac{K}{s(s+4)}$ is

$$G(z) = K\frac{0.158z}{(z-1)(z-0.368)}$$

The characteristic equation becomes

$$1+G(z)=0 \Rightarrow 1+K\frac{0.158z}{(z-1)(z-0.368)} = 0 \Rightarrow (z-1)(z-0.368)+K0.158z=0$$

Using $z = \dfrac{\lambda + 1}{\lambda - 1}$ to transform from the z plane to the λ plane. The characteristic equation becomes

The Routh array for the numerator is

$0.158K$	$(2.736 - 0.158K)$	0
1.264	0	
$(2.736 - 0.158K)$	0	

Thus, this system is unstable for

If, in the preceding example, the sampling rate is increased from 4 samples per seconds (T=1/4) to 10 samples per second (T=1/10), then the system would be unstable for $K \geq 42$. In general, making the sampling time shorter tends to make the system behave more like the corresponding continuous system.

Stability is improved as the sampling rate is increased.

Ex2: If a zero-order hold is included as shown in Fig.(2), find the value of K such that the system becomes unstable using Routh-Hurwitz criterion. (Use T=0.25 second).

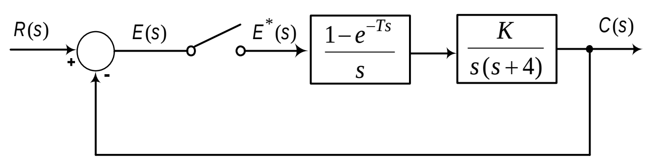


Figure (2) *Sampled-data system with a zero-order hold*

For $T = 0.25$, $G(z)$ is

$$G(z) = \frac{0.368K\,(z + 0.717)}{16\,(z-1)\,(z-0.368)}$$

The corresponding characteristic equation for this sampled-data system is

$$1 + G(z) = 0 \;\Rightarrow$$

$$\Rightarrow$$

Replacing z by $(\lambda+1)/(\lambda-1)$ so that Routh's criterion may be applied gives

$$0.0395K\,\lambda^2 + (1.264 - 0.033K)\,\lambda + (2.736 - 0.0065K) = 0$$

The Routh array is

$$\begin{array}{lll} 0.0395K & (2.736 - 0.0065K) & 0 \\ (1.264 - 0.033K) & 0 & \\ (2.736 - 0.0065K) & 0 & \end{array}$$

Thus, the system becomes unstable for

Without the zero-order hold, this system becomes unstable for . Thus one can conclude that

Stability is improved when the zero – order hold is included.

Jury's Test

Jury's test is a stability test which has some advantages over the Routh's test for continuous-data system. In general, given the polynomial in z,

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_2 z^2 + a_1 z + a_0 = 0 \tag{1}$$

where $a_0,\ a_1, \dots, a_n$ are real coefficients. Assuming that a_n is positive, or that it can be made positive by changing the signs of all coefficients, the following table is made:

Row	z^0	z^1	z^2	....	z^{n-k}	....	z^{n-1}	z^n
1	a_0	a_1	a_3	....	a_{n-k}	....	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	....	a_k	....	a_1	a_0
3	b_0	b_1	b_2	....	b_{n-k}	....	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	....	b_k	....	b_0	
5	c_0	c_1	c_2	....		c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	....		c_0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	....				
$2n-5$	p_0	p_1	p_2	p_3				
$2n-4$	p_3	p_2	p_1	p_0				
$2n-3$	q_0	q_1	q_2					

Note that the elements of the *(2k+2)th* row (*k=0,1,2,...*) consists of the coefficients of the (2k+1)th row are written in the reverse order. The elements in the table are defined as

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \qquad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \qquad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$

$$,\dots q_0 = \begin{vmatrix} p_0 & p_3 \\ p_3 & p_0 \end{vmatrix}, \qquad q_2 = \begin{vmatrix} p_0 & p_1 \\ p_3 & p_2 \end{vmatrix}$$

The necessary and sufficient conditions for the polynomial F(z) to have no roots on and outside the unit circle in the z-plane are:

$$\left.\begin{array}{l} F(1)>0 \\ F(-1)=\begin{cases} >0 & \text{n even} \\ <0 & \text{n odd} \end{cases} \\ \left.\begin{array}{l} |a_0|< a_n \\ |b_0|>|b_{n-1}| \\ |c_0|>|c_{n-2}| \\ |d_0|>|d_{n-3}| \\ \vdots \\ |q_0|>|q_2| \end{array}\right\} \quad (n-1)\ constaint s \end{array}\right\} \qquad (2)$$

For a second-order system, n=2, Jury's tabulation contains only one row. Therefore, the requirements listed in Eq.(2) are reduced to

$$F(1) > 0, \quad F(-1) < 0 \quad \text{and} \quad |a_0| < a_n$$

As in the Routh-Hurwitz criterion which is used for stability testing of linear continuous-data, occasionally the first element of a row or a complete row of the tabulation may be zero before the tabulation is scheduled to terminate. These cases are referred as singular cases. In Jury's tabulation a singular case is signified by either having the first and the last elements of a row be zero, or having a complete row of zeros.

The Singular Cases:

When some or all of the elements of a row in the Jury's tabulation are zero, the tabulation ends prematurely. This situation is referred to as the *singular case*. The singular case can be eliminated by expanding and contracting the unit circle infinitesimally, which is equivalent to moving the roots off the unit circle. The transformation for this purpose is

$$z=(1+\varepsilon)\,z \qquad (3)$$

where ε is a very small real number. When ε is a positive number in Eq.(3), the radius of the unit circle is expanded to $1+\varepsilon$, and when ε is negative, the radius of the unit circle is reduced to $1-\varepsilon$. This is equivalent to moving the roots slightly. The difference between the number of roots found inside (or outside) the unit circle when the unit circle is expanded or contracted by ε is the number of roots on the circle.

The transformation in Eq.(3) is actually ver easy to apply, since

$$(1 \pm \varepsilon)^n \cong (1 \pm n\varepsilon)\, z^n \qquad (4)$$

This means that the coefficient of z^n term is multiplied by $(1 \pm n\varepsilon)$.

Example1:

If the characteristic equation of a system is

$$F(z) = z^2 + z + 0.25 = 0$$

The first two conditions of Jury's test in Eq.(2) lead to

and

Since n=2 is even, these results satisfy the F(1)>0 and F(-1)<0 requirements for stability. Next, we tabulate the coefficients of F(z) according to Jury's test; we have

Since 2n-3=1, Jury's tabulation consists of only one row. The result is

and thus the system is stable, and all roots are inside the unit circle.

Example 2:

Consider the equation

$$F(z) = z^3 + 3.3z^2 + 3z + 0.8 = 0$$

which has roots at z=-0.5, -0.8, and -2.

From Jury's test, F(1)=8.1 and F(-1)=0.1. For odd n, since F(-1) is not negative, F(z) has at least one root outside the unit circle.

Example 3:

For the following characteristic equation

$$z^2 - z(1.48 - 0.025K) + (0.5026 + 0.0204K) = 0$$

Find the range of K for stability.

The first two conditions of Jury's test in Eq.(2) lead to

F(1)=

F(-1)=

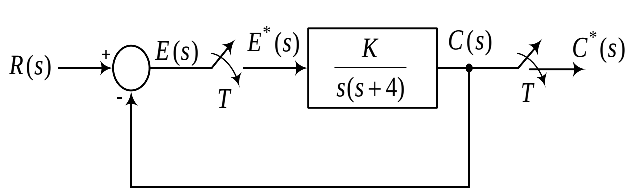
Since n=2 is even, these results satisfy the F(1)>0 and F(-1)<0 requirements for stability. Next, we tabulate the coefficients of F(z) according to Jury's test; we have

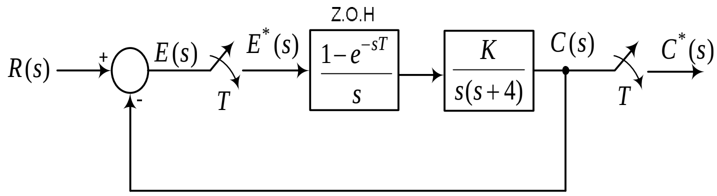
Since 2n-3=1, Jury's tabulation consists of only one row. The result is

$|a_0|$ =

Since K > 0, then the range of stability is and for

HW: For the following block diagrams, use Routh-Hurwitz criterion and Jury's test to find the range of K for stable system. (T=0.25 sec)




R(s)
+
-
E(s)
E*(s)
T
Z.O.H
1−e^{−sT} / s
K / s(s+4)
C(s)
T
C*(s)

Root Locus in the z-plane

As with the continuous systems, the root locus of a discrete system is a plot of the locus of the roots of the characteristic equation

$$1+GH(z)=0$$

in the z-plane as a function of the open-loop gain constant K. The closed-loop system will remain stable providing the loci remain within the unit circle.

Root Locus Construction Rules:

These are similar to those given in continuous systems.

- Starting points ($K=0$). The root loci start at the open-loop poles.
- Termination points ($K=\infty$). The root loci terminate at the open-loop zeros when they exist, otherwise at ∞.
- Number of distinct root loci (branches): This is equal to the order of the characteristic equations (or the number of poles of open loop transfer function).
- Symmetry of root loci: The root loci are symmetric about the real axis.
- Root locus locations on the real axis: A point on the real axis is part of the loci if the sum of the open-loop poles and zeros to the right of the point concerned is odd.
- Break away (in) points. The points at which a locus breaks away from (or break in) the real axis can be found by letting K as a function of z, taking the derivative of dK/dz and then setting the derivative equal to zero.
- Unit circle crossover: This can be obtained by determining the value of K for marginal stability using Jury test or Routh-Hurwitz criterion.

1. Root Locus without Zero Order Hold

<u>Ex:</u> Sketch the root locus for the diagram shown in Fig.(1)

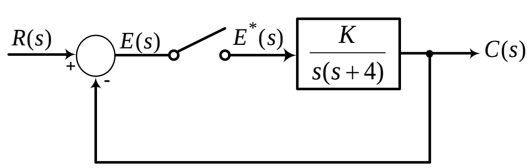


Figure (1) *Sample-data system*

The z-transform for the output $C(z)$ is

$$C(z)=\frac{G(z)}{1+G(z)}R(z)$$

The z-transformed characteristic equation is

$$1+G(z)=0$$

The partial fraction expansion for G(s) is

$$G(s)=\frac{K}{4}\left(\frac{1}{s}-\frac{1}{s+4}\right)$$

The corresponding z transform is

$$G(z)=\frac{K}{4}\left(\frac{z}{z-1}-\frac{z}{z-e^{-4T}}\right)=\left(\frac{K}{4}\right)\frac{z(1-e^{-4T})}{(z-1)(z-e^{-4T})}$$

For T=0.25 sec.

- Open-loop poles and zeros:

  Poles: $z=1$ and $z=0.368$

  Zeros: $z=0$

- Number of branches: Number of branches equals No. of poles=2.
- Root locus locations on the real axis: The root locus on the real axis lies between poles ($z=1$ and $z=0.368$) and to the left of zero (z=0).
- Break away and in points:

  The characteristic equation is

$$1+G(z)=1+K\frac{0.158\,z}{(z-1)(z-0.368)}=0$$

or

Then $$\frac{dK}{dz}=-\left(\frac{1}{0.158}\right)\frac{z(2z-1.368)-(z^2-1.368z+0.368)}{(z)^2}=0$$

or

To find the value of K at break away and in points, we use the magnitude condition:

The gain K at breakaway point:

$$K=\left|\frac{(z-1)(z-0.368)}{0.158z}\right|_{z=0.606}=\left[\frac{|(z-1)||(z-0.368)|}{|0.158z|}\right]_{z=0.606}$$

The gain K at break in point:

$$K = \left| \frac{(z-1)(z-0.368)}{0.158z} \right|_{z=-0.606}$$

- Crossing points of z-plane imaginary axis:

In general $z = a + jb$, and when the root locus crosses the imaginary axis of the z-plane, then the real part becomes zero, or $z = jb$. Substitute this value in the characteristic equation one can obtain:

$$z^2 - 1.368z + 0.368 + 0.158K\,z = 0$$
$$(jb)^2 - 1.368(jb) + 0.368 + 0.158K\,(jb) = 0$$

or

$$-b^2 - j1.368b + 0.368 + j0.158Kb = 0$$
$$\underbrace{(-b^2 + 0.368)}_{\mathrm{Re}\,al} + \underbrace{j(-1.368b + 0.158Kb)}_{\mathrm{Im}\,aginary} = 0$$

Two equations will be obtained:

$-b^2 + 0.368 = 0$ and $-1.368b + 0.158K\,b = 0$

From the first equation one can obtain the point of interception of root locus with the imaginary axis

$$-b^2 + 0.368 = 0 \;\Rightarrow\; b = \pm 0.606 \rightarrow z = \pm j0.606$$

Substitute the value of b at the second equation, the value of gain K at the imaginary axis becomes

$$-1.368b + 0.158K\,b = 0 \;\Rightarrow\; -1.368 \times 0.606 + 0.158K\,0.606 = 0 \rightarrow K = 8.658$$

Alternatively, one can use the magnitude condition to find the value of K at imaginary axis crossing points: (use either $z = j0.606$ or $z = -j0.606$)

$$K = \left| \frac{(z-1)(z-0.368)}{0.158z} \right|_{z=j0.606} = \left[\frac{|(z-1)||(z-0.368)|}{|0.158z|} \right]_{z=j0.606}$$

$$= \left[\frac{|(j0.606-1)||(j0.606-0.368)|}{|0.158j0.606|} \right]$$

- K for marginal stability: Using Routh-Hurwitz criterion (or Jury test), the value of K as the root locus crosses the unit circle into the unstable region is

$$K = 17.316$$

- Unit circle crossover: Inserting $K = 17.3$ into the characteristic equation

$$1 + G(z) = 1 + K\frac{0.158\,z}{(z-1)(z-0.368)} = 0 \Rightarrow 1 + 17.316 \times \frac{0.158\,z}{(z-1)(z-0.368)} = 0$$

$\Rightarrow\ z^2 + 1.367z + 0.368\ \Rightarrow$ The roots are $z = \pm 1$

- Angle of asymptotes

$$\lambda = \frac{(2n+1)180}{p-z} \qquad \text{n=0,1,2,3}$$

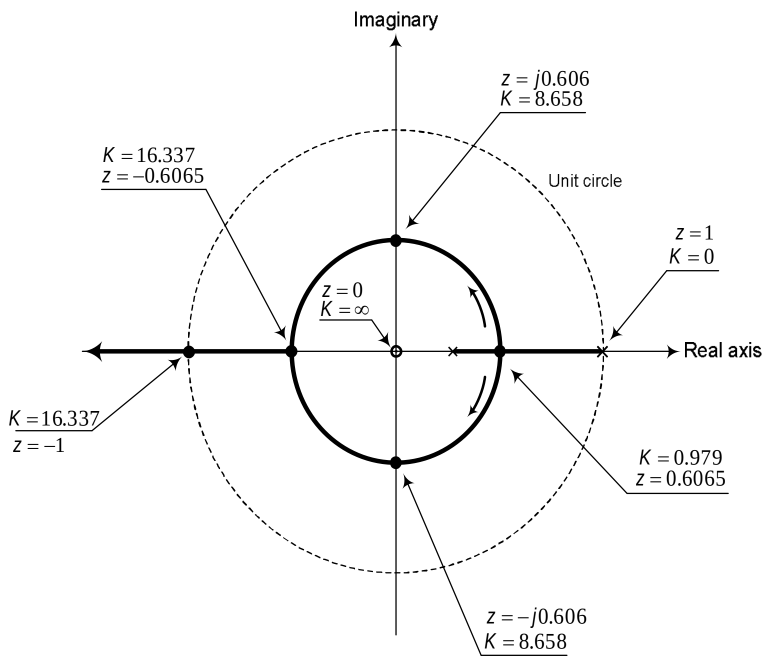
where p=number of poles and z is the number of zeros. Thus λ becomes

$$\lambda = 180$$

The real axis interception of the asymptotes is

$$\sigma_x = \frac{\sum_0^p z_p - \sum_0^z z_z}{p-z} = \frac{1 + 0.368 - 0}{2-1} = 1.368$$

The complete root-locus plot may now be constructed as shown in the following figure



Ex2: for the diagram shown in Fig.(2),

- Sketch the root locus for T=1/4 sec.
- Plot the response of the system to a unit step function and for K=4.

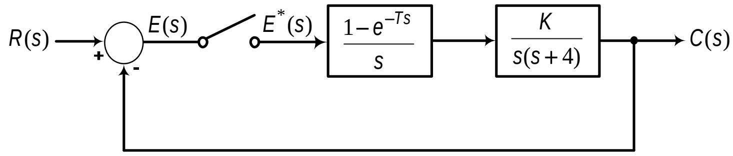


Figure (2) *Sample-data system*

For T=1/4, one can show that $G(z)$ has the form

$$G(z)=\frac{0.368K\ (z+0.717)}{16(z-1)(z-0.368)}$$

The z-transform closed-loop tranfer function

$$\frac{C(z)}{R(z)}=\frac{G(z)}{1+G(z)}$$

The characteristic equation of the above transfer function is

$$1+G(z)=0$$

or

$$z^2+(0.023K-1.368)z+(0.368+0.01649\,K)=0$$

- Open-loop poles and zeros:

Poles:

Zeros:

Number of branches: Number of branches equals No. of poles=2.

- Root locus locations on the real axis: The root locus on the real axis lies between poles ($z=1$ and $z=0.368$) and to the left of zero ($z=0.717$).
- Break away and in points:

The characteristic equation is

$$1+G(z)=1+K\frac{0.368\ (z+0.717)}{16\ (z-1)(z-0.368)}=0$$

or

Then

$$\frac{dK}{dz}=-(43.478)\ \frac{(z+0.717)(2z-1.368)-(z^2-1.368z+0.368)}{(z+0.717)^2}=0$$

or

$$(2z + 0.066z - 0.980) - (z^2 - 1.368z + 0.368) = 0 \Rightarrow z^2 + 1.434z - 1.348 = 0$$

Then

To find the value of K at break away and in points, we use the magnitude condition:

The gain K at breakaway point:

$$K = \left|(43.478)\ \frac{(z-1)(z-0.368)}{(z+0.717)}\right|_{z=0.647} = 43.478\left[\frac{|(z-1)||(z-0.368)|}{|(z+0.717)|}\right]_{z=0.647}$$

$$= 43.478\left[\frac{|0.647-1| \times |0.647-0.368|}{|0.647+0.717|}\right] = 3.139$$

The gain K at break in point:

$$K = 43.478\left|\frac{(z-1)(z-0.368)}{(z+0.717)}\right|_{z=-2.081} = 43.478\left[\frac{|(z-1)||(z-0.368)|}{|z+0.717|}\right]_{z=-2.081}$$

$$= 43.478\left[\frac{|-2.081-1| \times |-2.081-0.368|}{|-2.081+0.717|}\right] = 240.511$$

- Crossing points of z-plane imaginary axis:

In general $z = a + jb$, and when the root locus crosses the imaginary axis of the z-plane, then the real part becomes zero, or $z = jb$. Substitute this value in the characteristic equation one can obtain:

$$(jb)^2 + (0.023K - 1.368)\,jb + (0.368 + 0.01649\,K) = 0$$

or

Two equations will be obtained:

$(0.368 + 0.01649\,K - b^2) = 0$ and $b(0.023K - 1.368) = 0$

From the second equation one can determine the value of gain at the point of root-locus interception with the imaginary axis

$$b(0.023K - 1.368) = 0 \Rightarrow K = 59.478$$

Substitute the value of K into the first equation, the value of z at the imaginary axis becomes

$$(0.368 + 0.01649\,K - b^2) = 0 \quad \Rightarrow$$

Then, $z = \pm j1.161$ at the imaginary axis of the z-plane.

- K for marginal stability: Using Routh-Hurwitz criterion (or Jury test), the value of K as the root locus crosses the unit circle into unstable region is

$$K = 38.3$$

- Unit circle crossover: Inserting $K = 38.3$ into the characteristic equation

$$z^2 + (0.023 \times 38.3 - 1.368)z + (0.368 + 0.01649 \times 38.3) = 0$$

or

$$z^2 - 0.4871z + 0.999 = 0 \;\Rightarrow z = 0.2435 \pm j0.9693 \text{ (crossing points)}$$

- Angle of asymptotes

$$\lambda = \frac{(2n+1)180}{p-z} \qquad \text{n=0,1,2,3}$$

where p=number of poles and z is the number of zeros. Thus λ becomes

$$\lambda = 180$$

The real axis interception of the asymptotes is

The complete root-locus plot may now be constructed as shown in the figure below. Let it now be desired to determine the response of this system to a unit step function for the case in which K=4. It follows that

$$G(z) = \frac{0.368K\ (z+0.717)}{16(z-1)(z-0.368)} = \frac{0.092(z+0.171)}{z^2 - 1.368z + 0.367}$$

The z-transform closed-loop tranfer function

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.092\ (z+0.171)}{(z^2 - 1.368z + 0.368) + 0.092(z+0.717)}$$

Thus,

$$C(z) - 1.276\ z^{-1} C(z) + 0.434 z^{-2}\ C(z) = 0.092\ z^{-1} R(z) + 0.066\ \ z^{-2} R(z)$$

The corresponding recursive time difference equation is given by

The substitution of c(nT)=r(nT)=0 for n<0 and r(nT)=1 for K$\geq$0 yields the following values for c(nT) at the sampling instants:

c(0)=0

c(T) =

c(2T) = c(

c(3T)=c

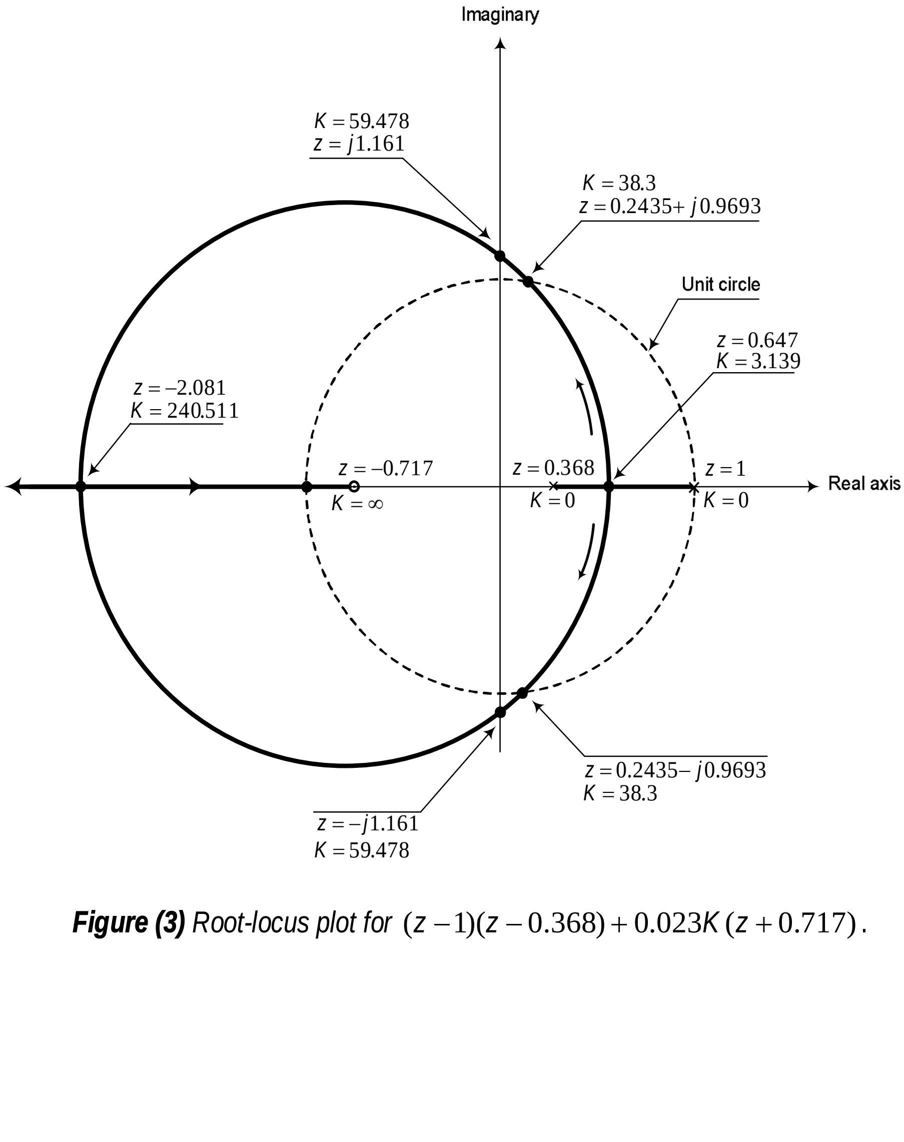


Figure (3) *Root-locus plot for* $(z-1)(z-0.368)+0.023K(z+0.717)$.

Design of Digital Control Systems with the Deadbeat Response

The design objectives of control systems can be classified as follows:

- A large number of control systems are designed with the objective that the responses of the systems should reach respective desired values as quickly as possible. This class of control systems is called minimum-time control systems, or time-optimal control systems.
- With reference to the previous design methods, one of the design objectives is to have a small maximum overshoot and a fast rise time in the step response.

In reality, the design principles discussed in the preceding sections involve the extension of the design experience acquired in the design of continuous-data control systems; e.g., phase-lag and phase-lead controllers, and the PID controllers.

However, since the digital controller has a great deal of flexibility in its configuration, one should be able to come up with independent methods not relying completely on the principles of design of continuous-data control system. We were perhaps amazed by what the PID controller could accomplish in the improvement on the system response for the digital control system, but can we do better?.

The answer is that in digital control system we may design the digital compensator $G_c(z)$ to obtain a response (output) with a finite settling time. The output response $c(kT)$ which reaches the desired steady-state value in a finite number of sampling intervals is called a ***deadbeat response***.

Ex1: The block diagram of a digital control system, shown in Fig.(1), is revisited. Again, the controlled process is represented by the transfer function

$$G_p(s) = \frac{10}{(s+1)(s+2)}$$

Try to find a controller with the objective to cancel all poles and the zeros of the process and then add a pole at $z = 1$.

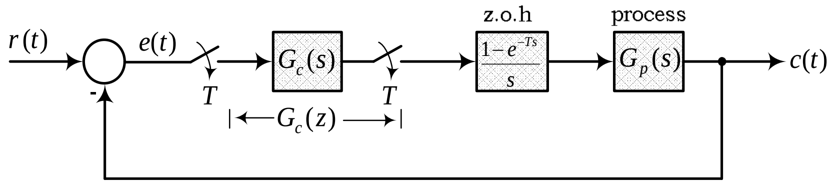


Figure (1) A digital control system with a digital controller.

The open-loop pulse transfer function of the uncompensated system is

$$G_{zoh}G_p(z) = (1 - z^{-1})\ \mathcal{Z}\left[\frac{10}{s\,(s+1)\,(s+2)}\right] = \frac{0.0453(z+0.904)}{(z-0.905)\,(z-0.819)}$$

The pulse transfer function of the suggested digital controller be

$$G_c = \frac{(z-0.905)\,(z-0.819)}{0.0453(z-1)\,(z+0.904)}$$

The open-loop transfer function of the compensated system now simply becomes

$$G_c(z)\,G_{zoh}G_p(z) = \frac{1}{z-1}$$

The corresponding closed-loop transfer function is

$$\frac{C(z)}{R(z)} = \frac{1}{z}$$

Thus, for a unit step input, the output transform is

$$C(z) = \frac{1}{z-1} = z^{-1} + z^{-2} + z^{-3} + \cdots$$

The following points have to be highlighted:

- The output response $c(kT)$ reaches the desired steady-state value in one sampling period and stays at that value thereafter.
- In reality, however, it must be kept in mind that the true judgement on the performance should be based on the behavior of $c(t)$. In general, although $c(kT)$ may exhibit little or no overshoot, the actual response $c(t)$ may have oscillations between the sampling instants.
- For the present system, since the sampling period $T = 0.1\sec$ is much smaller than the time constants of the controlled process, it is expected that $c(kT)$ gives a fairly accurate description $c(t)$.
- Thus, it is expected that the digital controller will produce a unit-step response that reaches its steady-state value of 0.1 sec, and there should be little or no ripple in between the sampling instants.
- This type of response is referred to **deadbeat response**.

However, the design based on the deadbeat response still has the following limitations and criteria:

- The deadbeat response is obtainable only under the ideal condition that the cancellation of poles and zeros as required by the design is exact. In practice, the uncertainty of the poles and zeros of the controlled process, due to the approximations required in the modeling of the process, and the restrictions in the realization of the controller transfer function by a digital computer or processor, would make a deadbeat response almost impossible to achieve exactly.
- The system must have zero steady-state error at the sampling instants for the specified reference input signal.
- The response time defined as the time required to reach the steady state should be a minimum.
- The digital controller $G_c(s)$ must be physically realizable.

The closed-loop pulse transfer function of the digital-controlled system shown in Fig.(1) is

$$\frac{C(z)}{R(z)} = M(z) = \frac{G_c(z)G(z)}{1+G_c(z)G(z)} \tag{1}$$

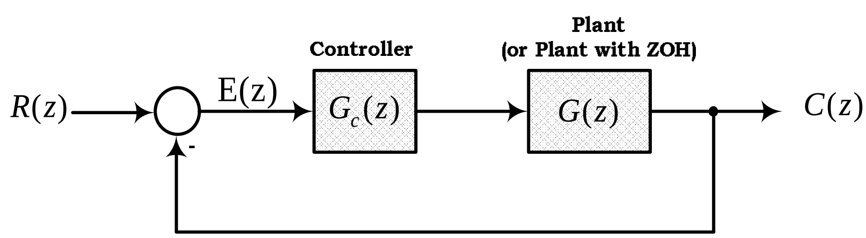


Figure (2) Rotational Dynamics of a Satellite (Pure inertia)

Solving for $G_c(z)$ from Eq.(1), we have

$$G_c(z) = \frac{1}{G(z)} \frac{M(z)}{1-M(z)} \tag{2}$$

<u>Steady-state error</u>

The z-transform of the error signal is written

$$\begin{aligned} E(z) &= R(z) - C(z) \\ &= R(z)\,[1-M(z)] = \frac{R(z)}{1+G_c(z)G(z)} \end{aligned} \tag{3}$$

Let the z-transform of the input be described by the function

$$R(z) = \frac{A(z)}{(1-z^{-1})^N} \tag{4}$$

where N is a positive integer, and $A(z)$ is a polynomial in z^{-1} with no zeros at $z=1$. For example, for a unit-step function input, $A(z)=1$ and $N=1$; for a unit-ramp function input, $A(z)=Tz^{-1}$ and $N=2$, etc. In general, $R(z)$ of Eq.(4) represents inputs of type t^{N-1}. For zero steady-state error,

$$\underbrace{\lim}_{k\to\infty} e(kT) = \underbrace{\lim}_{z\to 1}(1-z^{-1})\ E(z)$$

$$= \underbrace{\lim}_{z\to 1}(1-z^{-1})\ \frac{A(z)[1-M(z)]}{(1-z^{-1})^N} = 0 \tag{5}$$

Since the polynomial $A(z)$ does not contain any zeros at $z=1$, it is necessary condition for the steady-state error to be zero is that $1-M(z)$ must contain the factor $(1-z^{-1})^N$. Thus, $1-M(z)$ should have the form

$$1-M(z) = (1-z^{-1})^N\ F(z) \tag{6}$$

where $F(z)$ is a polynomial of z^{-1}

$$F(z) = 1 + z^{-1} + z^{-2} + \cdots + z^{-f} \tag{7}$$

where f denote the largest order of $F(z)$, which is selected to achieve the realizability of the controller $G_c(z)$. Solving for $M(z)$ in the Eq.(6) one can have

$$M(z) = \frac{z^N - (z-1)^N F(z)}{z^N} \tag{8}$$

Since $F(z)$ is a polynomial in z^{-1}, it has only poles at $z=0$. Therefore, Eq.(8) clearly indicates that the characteristic equation of the system with zero steady-state error is of the form

$$z^p = 0 \tag{9}$$

where p is a positive integer $\geq N$.

Substitution of Eq.(5) into Eq.(3), the z-transform of the error is written as

$$E(z) = A(z)F(z) \tag{10}$$

Since $A(z)$ and $F(z)$ are both polynomial of z^{-1}, $E(z)$ in Eq.(9) will have a finite number of terms in terms in its power-series expansion in inverse powers of z. Thus, when the characteristic equation of a digital control system is of the form of Eq.(9), that is, when the characteristic equation roots are all at $z=0$, the error signal will go to zero in finite number of sampling periods.

Physical realizability considerations

Equation (8) indicates that the design of a digital control system with the deadbeat response for a given input requires first the selection of the function $F(z)$. Once $M(z)$ is determined, the transfer function of the digital controller is obtained from Eq.(2). However, the physical realizability requirement on $G_c(z)$ and the fact that $G(z)$ is transfer function of a physical process put constrains on the closed-loop transfer function $M(z)$. In general, let $G(z)$ and $M(z)$ be expressed by the following series expansions:

$$\left.\begin{aligned} G(z) &= g_o z^{-n} + g_1 z^{-n-1} + g_2 z^{-n-2} + \cdots \qquad n \ge 0 \\ M(z) &= m_o z^{-k} + m_1 z^{-k-1} + m_2 z^{-k-2} + \cdots \qquad k \ge 0 \end{aligned}\right\} \tag{11}$$

Substituting the last two equations in Eq.(2), one can have

$$\begin{aligned} G_c(z) &= \frac{(m_o z^{-k} + m_1 z^{-k-1} + m_2 z^{-k-2} + \cdots)}{(g_o z^{-n} + g_1 z^{-n-1} + g_2 z^{-n-2} + \cdots)(1 - m_o z^{-k} + m_1 z^{-k-1} + m_2 z^{-k-2} + \cdots)} \\ &= d_o z^{-(k-n)} + d_1 z^{-(k-n+1)} + d_2 z^{-(k-n+2)} + \cdots \end{aligned} \tag{12}$$

✍ Thus, for $G_c(z)$ to be physically realizable, $k \ge n$; i.e., the lowest power of the series expansion of $M(z)$ in inverse powers of z must be at least as large as that of $G(z)$. Once the minimum requirement on $M(z)$ is established, for a specific input, $F(z)$ must be chosen according to Eq.(6) to satisfy this requirement.

The relations between the basic form of $M(z)$ and the type of input for a deadbeat response are determined from Eq.(8) and tabulated in Table (1).

Table (1)

Input function	N	M(z)	M(z) with F(z)=1
Step input $u(t)$	1	$1-(1-z^{-1})F(z)$	z^{-1}
Ramp input $t\ u(t)$	2	$1-(1-z^{-1})^2 F(z)$	$2z^{-1}-z^{-2}$
Ramp input $t^2\ u(t)$	3	$1-(1-z^{-1})^3 F(z)$	$3z^{-1}-3z^{-2}+z^{-3}$

In fact, there does not seem to be any objection to selecting $F(z)=1$ for all types of input. Thus, the results in Table (1) indicate that

- When the input is a step function input, the minimum time for the error to go to zero is one sampling period.
- For a ramp input, it takes two sample periods for the error to be reduced to zero.
- The minimum number of sampling periods for the error, due to a parabolic input, to diminish is three.

Another difficulty is revealed by referring to Eq.(2), when $M(z)$ any one of the forms given in Table (1). Since the highest power term in $M(z)$ is z^{-1}, $M(z)/[1-M(z)]$ will always have one more pole than zero. Then in order that $G_c(z)$ is physically realizable transfer function, $G(z)$ must have at most one more pole than zero. Of course, $G(z)$ can not have more zeros than poles. For example, for a step input, $M(z)=z^{-1}$, Eq.(2) gives

$$G_c(z)=\frac{1}{G(z)}\frac{1}{z-1}$$

Thus, the condition on $G(z)$ given above is arrived at. The conclusion is that when $G(z)$ has more than one pole than zeros, $F(z)$ can not be simply 1.

Ex2: Using the above analysis, repeat the design of the digital controller in the previous example to give deadbeat response in one sampling interval to unit step input.

Using Eq.(2), the controller transfer function can be written as

$$G_c(z)=\frac{1}{G(z)}\frac{M(z)}{1-M(z)}=\frac{1}{\left(\dfrac{0.0453(z+0.904)}{(z-0.905)(z-0.819)}\right)}\frac{M(z)}{1-M(z)}$$

One can see that he transfer function $G(z)$ has one more pole than zero. Thus, for the digital controller $G_c(z)$ to be a physically realizable transfer function, then, one can choose $M(z)$ to be z^{-1} and $F(z)$ to be unity. The controller transfer function becomes

$$G_c=\frac{(z-0.905)(z-0.819)}{0.0453(z-1)(z+0.904)}$$

Ex3: Consider that the controlled process of the digital control system shown in Fig.(2) is described by

$$G(z)=\frac{1}{z^2-z-1}$$

A digital controller is to be designed so that a deadbeat response is obtained when the unit input is a unit step input.

Since the transfer function $G(z)$ has two more poles than zeros, we can not choose $M(z)$ to be z^{-1}, since it will lead to a physically unrealizable $G_c(z)$. Let us try $M(z)=z^{-2}$. Then,

$$G_c(z)=\frac{1}{G(z)}\frac{M(z)}{1-M(z)}=\frac{1-z^{-1}-z^{-2}}{z^{-2}}\frac{z^{-2}}{1-z^{-2}}=\frac{1-z^{-1}-z^{-2}}{1-z^{-2}}$$

which is a physically realizable transfer function. In this case, the function $F(z)$ is given by Eq.(6)

$$F(z)=\frac{1-M(z)}{(1-z^{-1})^N}=\frac{1-M(z)}{(1-z^{-1})}=\frac{(1-z^{-2})}{(1-z^{-1})}=\frac{(1-z^{-1})(1+z^{-1})}{(1-z^{-1})}=1+z^{-1}$$

Ex4: Consider the digital controlled system of a simple satellite rotational dynamics shown in Fig.(3). The figure shows pure inertia plant driven by zero-order hold.

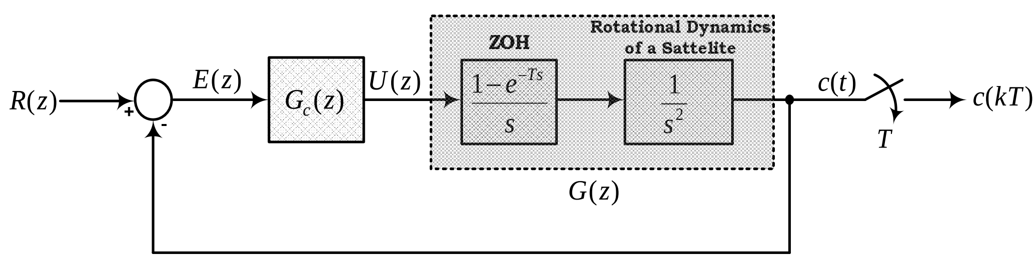


Figure (3) Rotational Dynamics of a Satellite (Pure inertia)

The overall transfer function by

$$G(z)=(1-z^{-1})\ \mathcal{Z}\left[\frac{G(s)}{s}\right]\Rightarrow\quad G(z)=(1-z^{-1})\ \mathcal{Z}\left[\frac{1}{s^3}\right]$$

or

$$G(z)=\frac{T^2}{2}\left[\frac{z^{-1}(1+z^{-1})}{2(1-z^{-1})^2}\right]$$

Letting $T = 0.1\sec$, then

$$G(z) = \frac{1}{200}\left[\frac{z^{-1}(1+z^{-1})}{2(1-z^{-1})^2}\right]$$

Applying Eq.(2), one can get

$$G_c(z) = \frac{1}{G(z)}\left[\frac{z^{-1}}{1-z^{-1}}\right] = \frac{200(1-z^{-1})^2}{z^{-1}(1+z^{-1})}\frac{z^{-1}}{1-z^{-1}}$$

and upon simplification, the above expression becomes

$$G_c(z) = \frac{200(1-z^{-1})}{1+z^{-1}} = \frac{U(z)}{E(z)}$$

Substitute the expression of the above controller into the overall pulse transfer function $M(z)$, one can obtain the following one delay closed loop pulse transfer function

$$M(z) = \frac{C(z)}{R(z)} = \frac{G_c(z)G(z)}{1+G_c(z)G(z)} = \frac{1}{z}$$

The block diagram representing the above pulse transfer function is shown in Fig.(4).

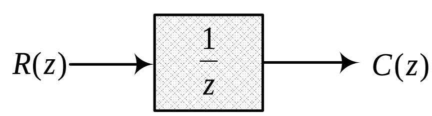


Figure (4) Simple delay element

Cross-multiplication of the controller transfer function $G_c(z)$, one can get the difference equation for the control algorithm

$$u(kT) = 200\left[e(kT) - e(kT-T)\right] - u(kT-T)$$

This algorithm has to be programmed into the digital controller. Figure (5) shows the unit step input response and control effort for both continuous and sampled forms. It is clear from the figure that the output reaches steady-state value (with zero error) in one sample time.

There are several possible problems which commonly occur in finite settling time (deadbeat) response of the studied example:

- The first is that in order to get 1-step settling time there is an excessive overshoot and sustained oscillation present in the continuous-time response.

✍ The second is that very high control efforts are required and these efforts could cause saturation problems at the output of the digital-to-analogue (D/A) converter or at best require high-power elements to generate the continuous-time control effort $u(t)$.

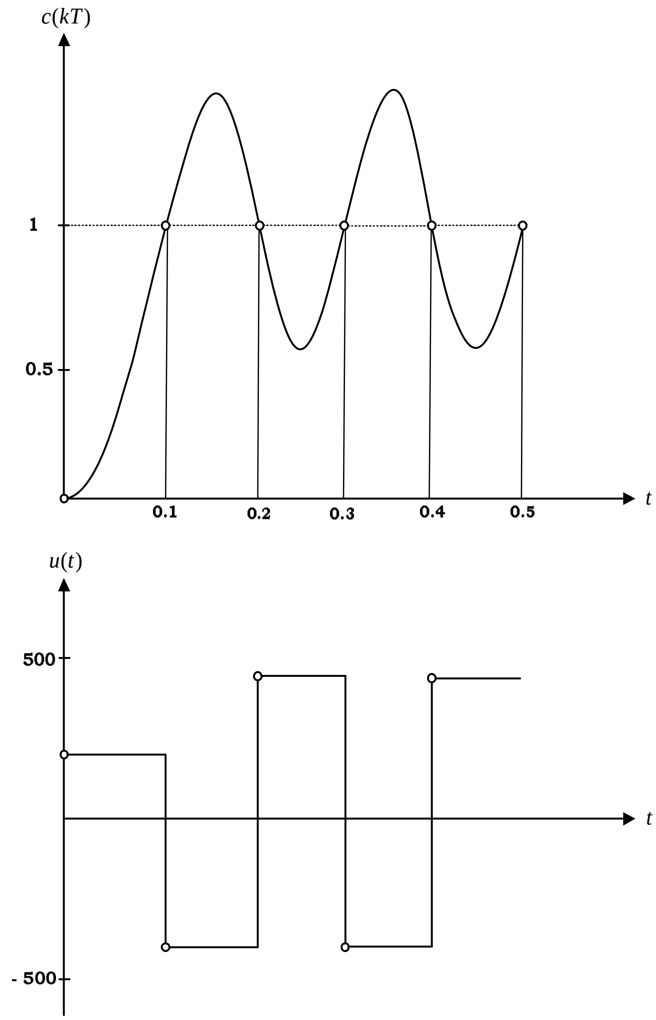


Figure (5) Response of inertial system and control effort for minimum prototype(deadbeat) controller

Ex5: Let us consider the settling of the temperature control of the thermal system to ramp input. The pulse transfer function of plant and zero order hold for a sampling interval $T = 0.25$ sec is

$$G(z) = \frac{0.025\,(z + 0.816)}{(z - 0.925)(z - 0.528)}$$

we would like to have this system follow the sampled ramp input which is

$$r(kT) = 4(kT)$$

If we apply the expression (6) for N=2, one can get a discrete controller transfer function of

$$G_c(z) = \frac{40\,(2z^{-1} - 3.96\,z^{-2} + 2.486\,z^{-3} - 0.503\,z^{-4})}{z^{-1} - 1.184\,z^{-2} - 0.632\,z^{-3} + 0.816\,z^{-4}} = \frac{U(z)}{E(z)}$$

which yields a difference equation for the control algorithm:

$$\begin{aligned} u(kT) &= 1.184u(kT-T) + 0.632u(kT-2T) - 0.816u(kT-3T) \\ &\quad + 40\left[2e(kT) - 3.96e(kT-T) + 2.486e(kT-2T) - 0.503e(kT-3T)\right] \end{aligned}$$

The result of this type of algorithm for a ramp refernce input is illustrated in Fig.(6).

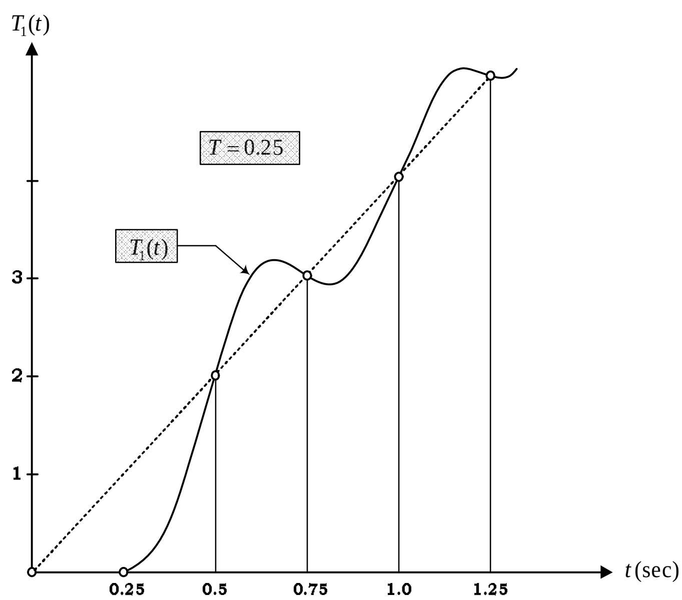


Figure (6) Response of thermal system deadbeat controller to ramp input

The forgoing examples have illustrated the algorithm for generation of minimal prototype systems which settle to polynomial-type input functions in one step than the order of the polynomial input. There are, however, some problems associated with such control algorithms:

- They require excessively high control efforts $u(t)$.
- As result of these high control efforts, continuous-time plants will tend to oscillate violently between sampling intervals.

These two problems make minimal prototype (deadbeat) systems not nearly so desirable and one might think that they are just of academic interest.

Steady State Error

An important characteristic of a control system is its ability to follow, or track, certain inputs with a minimum of error. The control system designer attempts to minimize the system error to certain anticipated inputs. In this section the effects of the system transfer characteristics on the steady-state system errors are considered.

Consider the system of Fig.(1). The signal $e(t)$ is defined as the error; that is,

$$e(t) = r(t) - b(t)$$

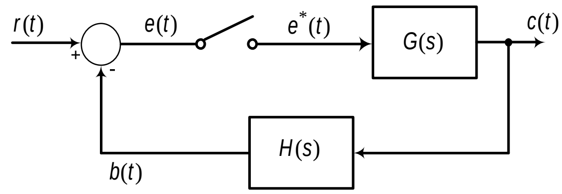


Figure (1) *A digital control system*

Since it is difficult to describe $e(t)$ in a digital control system, the sampled error $e^*(t)$ or the error at the sampling instants $e(kT)$ is usually used. Thus, the steady-state error at the sampling instants is defined as

$$E_{SS}^* = \lim_{k\to\infty} e(kT) = \lim_{t\to\infty} e^*(t)$$

Using the z-transform, *the final value theorem* leads to

(1)

For the system shown in Fig.(1), the z-transform of the error signal is written

$$E(z) = \frac{R(z)}{1+GH(z)}$$

Substituting the last equation into Eq.(1), we have

(2)

This expression shows that the steady state error depends on the reference input $R(z)$, as well as the loop transfer function $GH(z)$. In the following, three basic types of input signals will be considered: step function, ramp function and parabolic function.

❑ Steady State Error due to a Step Function input:

Let the reference input to the system of Fig.(1) be a step function of magnitude h. The z-transform of $r(t)$ is

$$R(z) = h\frac{z}{z-1}$$

Substituting the last equation into Eq.(2), we have

(3)

Let the step-error constant be defined as

$$K_p = \lim_{z \to 1} GH(z)$$

Then Eq.(3) becomes

$$E_{ss}^* = \frac{h}{1+K_p}$$

Thus, for the steady-state error due to a step function input to be zero, the step-error constant K_p must be infinite. This implies that the transfer function $GH(z)$ must have at least one pole at z=1.

- <u>Steady State Error due to a Ramp Function input:</u>

For a ramp function, $r(t) = ht$, the z-transform of $r(t)$ is

$$R(z) = \frac{hT\,z}{(z-1)^2}$$

Substitute the previous equation into Eq.(2), we have

(4)

Let the ramp-error constant be defined as

$$K_v = \frac{1}{T} \lim_{z \to 1} \left[(z-1)\, GH(z) \right] \qquad (5)$$

then, Eq.(4) becomes

$$E_{ss}^* == \frac{h}{K_v} \qquad (6)$$

The ramp-error constant K_v is meaningful only when the input to the system is a ramp function. Again, Eq.(6) is valid only if the function after the limit sign in Eq.(2) does not have any poles on or outside the unit circle $|z| = 1$. This means that the closed-loop digital control system must at least be asymptotically stable.

Equation (6) shows that in order for E_{ss}^* due to a ramp function input be zero, K_v must equal infinity. From Eq.(5) we see that this is equivalent to the requirement of $(z-1)GH(z)$ having at least one pole at z=1, or $GH(z)$ having two poles at z=1.

- <u>Steady State Error due to a Parabolic Function input:</u>

For a parabolic function, $r(t) = h\frac{t^2}{2}$, the z-transform of $r(t)$ is

$$R(z) = \frac{hT^2 z(z+1)}{2(z-1)^3}$$

From Eq.(2), the steady-state error at the sampling instants is written as

$$E_{ss}^* = \frac{T^2}{2} \lim_{z \to 1} \frac{h(z+1)}{(z-1)^2 [1 + GH(z)]}$$

or

(7)

Now, let the parabolic-error constant be defined as

$$K_a = \frac{1}{T^2} \lim_{z \to 1} [(z-1)^2 \; GH(z)] \tag{8}$$

Then, Eq.(7) becomes

$$E_{ss}^* = \frac{h}{K_a} \tag{9}$$

In a similar manner we must point out that the parabolic-error constant is associated only with the parabolic function input, and should not be used with any of the other types of inputs.

Effects of Sampling on the Steady-State Error:

If the open-loop transfer function of Fig.(1) is of the following form:

$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{s^j (1+T_1 s)(1+T_2 s)\cdots(1+T_n s)} \tag{10}$$

where the T's are nonzero real or complex constants, the type of the system is equal to j. The error constants for the continuous-data system are defined as

Step-error constant: $K_p = \lim_{s \to 0} G(s)H(s)$

Ramp-error constant: $K_v = \lim_{s \to 0} sG(s)H(s)$

Parabolic-error constant: $K_a = \lim_{s \to 0} s^2 G(s)H(s)$

According to the above equations, one can easily conclude that, for instance, a type-0 system will have a constant steady-state error due to a step function input, and infinite error due to all higher-order inputs. A type-1 system (j=1) will have a zero steady-state error due to a step-function input, a constant error due to a ramp function input, and infinite error due to all higher-order inputs. Table (1) summarizes the relationships between the system type, and the error constants for the continuous-data systems.

Table (1)

Type of System	K_p	K_v	K_a
0	K	0	0
1	∞	K	0
2	∞	∞	K

We will evaluate the error constants of digital control system shown in Fig.(1) for the cases of j=0,1 and 2 as follows:

- Type 0 (j=0)

  In this type j=0 and Eq.(10) becomes

$$G(s)H(s) = \frac{K\,(1+T_a\,s)(1+T_b\,s)\cdots(1+T_m\,s)}{(1+T_1\,s)(1+T_2\,s)\cdots(1+T_n\,s)}$$

where we assume that the open loop transfer function has more pole than zeros. The z transform of $G(s)H(s)$ is

(11)

Performing partial fraction expansion to the function inside the bracket in the last equation, we have

$$GH(z) = Z\left\{\frac{K_1}{(1+T_1 s)} + \frac{K_2}{(1+T_1 s)} + \cdots + \frac{K_n}{(1+T_n s)}\right\}$$

$$= \{\textit{Terms with nonzero poles}\}$$

It is important to note that the terms due to the nonzero poles do not contain the term $(z-1)$ in the denominator. Thus, the step-error constant is

$$K_p = \lim_{z\to 1} GH(z) = \lim_{z\to 1}\left[\textit{terms with nonzero poles}\right] = cons\tan t$$

Substituting Eq.(11) into the ramp-error constant of Eq.(8), we get

$$K_v = \frac{1}{T}\lim_{z\to 1}\,(z-1)\,GH(z) = \frac{1}{T}\lim_{z\to 1}(z-1)\left[\textit{terms with nonzero poles}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z-1)}{T}\right]\lim_{z\to 1}\left[(\textit{terms with nonzero poles}\right] = \lim_{z\to 1}\left[\frac{(z-1)}{T}\right]\left[cons\tan t\right] = 0$$

Similarly,

$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2\, GH(z) = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2\left[\textit{terms with nonzero poles}\right]$$
$$= \lim_{z\to 1}\left[\frac{(z-1)^2}{T^2}\right]\lim_{z\to 1}\left[\textit{terms with nonzero poles}\right] = \lim_{z\to 1}\left[\frac{(z-1)^2}{T^2}\right]\left[cons\tan t\right] = 0$$

- Type 1 (j=1)

  In this type j=1 and Eq.(10) becomes

$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{s(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)}$$

The z transform of $G(s)H(s)$ is

(12)

Performing partial fraction expansion to the function inside the bracket in the last equation, we have

$$GH(z) = Z\left\{\frac{K}{s} + \textit{terms due to the nonzero poles}\right\}$$
$$GH(z) = \left\{\frac{Kz}{z-1} + \textit{terms due to the nonzero poles}\right\}$$

Thus, the step-error constant is

$$K_p = \lim_{z\to 1} GH(z) = \lim_{z\to 1}\left[\frac{Kz}{z-1} + \textit{terms due to the nonzero poles}\right] = \left[\infty + value\right] = \infty$$

Substituting Eq.(12) into the ramp-error constant of Eq.(8), we get

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)\, GH(z) = \frac{1}{T}\lim_{z\to 1}(z-1)\left[\frac{Kz}{z-1} + \textit{terms due to the nonzero poles}\right]$$
$$= \lim_{z\to 1}\left[\frac{Kz}{T} + \frac{(z-1)}{T}(\textit{terms due to the nonzero poles})\right] = \frac{K}{T}$$

Similarly,

$$= \lim_{z\to 1}\left[\frac{Kz(z-1)}{T^2} + \frac{Kz(z-1)^2}{T^2}(\textit{terms due to the nonzero poles})\right] = 0$$

- Type 2 (j=2)

  For a type-1 system j=1, Eq.(11) becomes

$$GH(z) = Z\left[\frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{s^2(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)}\right] \tag{13}$$

$$GH(z) = Z\left\{\frac{K}{s^2} + \frac{K_1}{s} + \textit{terms due to the nonzero poles}\right\}$$

Then, the step-error constant is

$$K_p = \lim_{z\to 1} GH(z) = \lim_{z\to 1}\left[\frac{KTz}{(z-1)^2} + \frac{K_1 z}{z-1} + \textit{terms due to the nonzero poles}\right] = \infty$$

The ramp-error constant is

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)\,GH(z) = \frac{1}{T}\lim_{z\to 1}(z-1)\left[\frac{KTz}{(z-1)^2} + \frac{K_1 z}{z-1} + \textit{terms due to the nonzero poles}\right]$$

The parabolic-error constant

$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2\,GH(z) = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2\left[\frac{KTz}{(z-1)^2} + \frac{K_1 z}{z-1} + \textit{terms due to the nonzero poles}\right]$$

$$= \lim_{z\to 1}\left[\frac{Kz}{T} + \frac{K_1 z(z-1)}{T^2} + \frac{(z-1)^2}{T^2}(\textit{terms due to the nonzero poles})\right] = \left[\frac{K}{T} + 0 + 0\right] = \frac{K}{T}$$

One can summarize the above in the following Table

Table (1)

Type of System	K_p	K_v	K_a
0	constant	0	0
1	∞	K/T	0
2	∞	∞	K/T

It would seem that K_v and K_a all depend on the sampling period T

<u>Ex1</u>: Calculate the steady-state errors for the system of Fig(1), in which the open-loop transfer function is given as

$$G(s) = \left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{K}{s(s+1)}\right)$$

Thus

$$G(z) = K\, Z\left[\frac{1-e^{-Ts}}{s^2(s+1)}\right] = K\frac{(z-1)}{z}\, Z\left[\frac{1}{s^2(s+1)}\right]$$

Since $G(z)$ has one pole at z=1, the steady state error to unit step is zero, and to ramp input is $1/K$ provided that the system is stable.

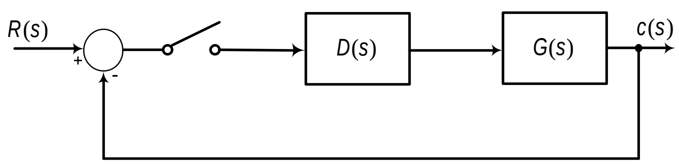
<u>Ex2</u>: Consider the system of Fig.(1), where $H(s)=1$ and

$$G(z) = \frac{1-e^{-T}}{z-e^{-T}}$$

Suppose that the design specification for this system requires that the steady state error to a unit ramp input be less that 0.01. Thus, it is necessary that the open-loop transfer function have a pole at z=1. Since the plant does not contain a pole at z=1, a digital compensator of the form

$$D(z) = \frac{K_i z}{z-1} + K_p$$

will be added to produce the resultant system shown in figure below.



The compensator, called a PI or proportional-plus-integral compensator, is of a form commonly used to reduce steady-state errors. Employing the expressions above for $D(z)$ and $G(z)$, we see that

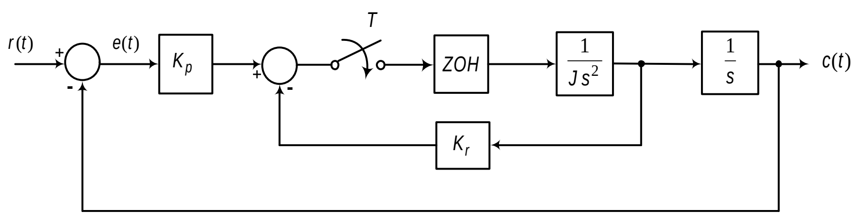
$$K_v = \frac{1}{T}\lim_{z\to 1}\left[\frac{(K_i+K_p)z-K_p}{(z-1)}\right]\left[\frac{1-e^{-T}}{z-e^{-T}}\right] = \frac{K_i}{T}$$

Thus K_i must equal $(100T)$ for the required steady-state error, provided that the system is stable. The latter point is needed an important consideration since the error analysis is meaningful unless the stability is guaranteed.

HW1: If a zero-order hold is included immediately after the sampler in the digital control system of Figure (1), then

- Follow the same above argument, show that the step, ramp and parabolic error constants are the same as given in Table (1) for continuous system.
- Do these error constants depend on the sampling period T? Why?

HW2: For the simplified digital control system in the figure shown below, find the step, ramp and parabolic error constants. Express the results in terms of the system parameters.



State Variable of Discrete Systems

In general, the analysis and design of linear systems may be carried out by one of the two major approaches:

- ☞ One approach relies on the use of Laplace and z-transforms, transfer functions, block diagrams or signal flows.
- ☞ The other method, which is synonymous with modern control theory, is the state variable technique. The fact is that a great majority of modern design techniques are based on the state variable formulation and modeling of the system.

In a broad sense, the state variable representation has the following advantages, at least in digital control system studies, over the conventional transfer function method.

- ☞ The state variable formulation is natural and convenient for computer solutions.
- ☞ The state variable approach allows a unified representation of digital systems with various types of sampling schemes.
- ☞ The state variable method allows a unified representation of single and multiple variable systems.
- ☞ The state variable method can be applied to certain types of nonlinear and time-varying systems.

In the state variable formulation a continuous-data system is represented by a set of first-order differential equations, called state equations. For a digital with all discrete-data components, the state equations are first-order difference equations.

State Equations of Continuous-Data System:

Consider that a continuous-data system with m inputs and r outputs as shown in Fig.(1) is characterized by the following set of n first-order differential equations, called state equations

$$\frac{d x_i(t)}{dt} = f_i\left[x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), u_m(t), t\right] \quad (i = 1,2,\ldots,n) \qquad (1)$$

where $x_1(t), x_2(t), \ldots, x_n(t)$ are the state variables, $u_1(t), u_2(t), \ldots, u_m(t)$ are the input variables, and f_i denotes the ith functional relationship. In general, f_i can be linear or nonlinear.

The r outputs of the system are related to the state variables and the inputs through the output equations which are of the form,

$$y_k(t) = h_k\left[x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), u_m(t), t\right] \quad (k = 1,2,\ldots,r) \tag{2}$$

Similar remarks can be made for h_k as for f_i.

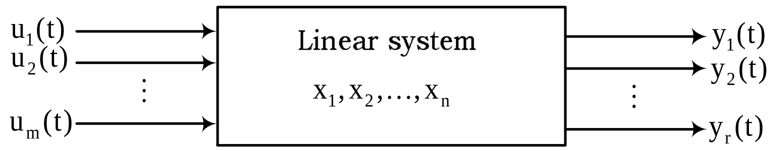


Figure (1) A linear system with m inputs, r outputs and n state variables

It is customary to write the dynamic equations in vector-matrix form:

State equation: $$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}\,[\mathbf{x}(t), \mathbf{u}(t), t] \tag{3}$$

Output equation: $$\mathbf{y}(t) = \mathbf{h}\,[\mathbf{x}(t), \mathbf{u}(t), t] \tag{4}$$

where $\mathbf{x}(t)$ is $n \times 1$ column matrix, and is called the state vector, that is

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{5}$$

The input matrix, $\mathbf{u}(t)$, is $m \times 1$ column matrix, and

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \tag{6}$$

The output vector, $\mathbf{y}(t)$, is defined as

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_r(t) \end{bmatrix} \tag{7}$$

which is a $r \times 1$.

If the system is linear but has time-varying elements, the dynamic equations Eq.s (3) and (4) are written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) \tag{8}$$
$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t) \tag{9}$$

where $\mathbf{A}(t)$ is $n \times n$ square matrix, $\mathbf{B}(t)$ is $n \times m$, $\mathbf{C}(t)$ is $r \times n$ and $\mathbf{D}(t)$ is $r \times m$. All the elements of these coefficient matrices are considered to be continuous functions of time t.

If the system is linear and time-invariant, Eq.s (8) and (9) are of the form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{10}$$
$$\mathbf{y} = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \tag{11}$$

The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ now all contain constant elements. The block diagram representing Eq.s (10) and (11) is shown in Fig.(2)

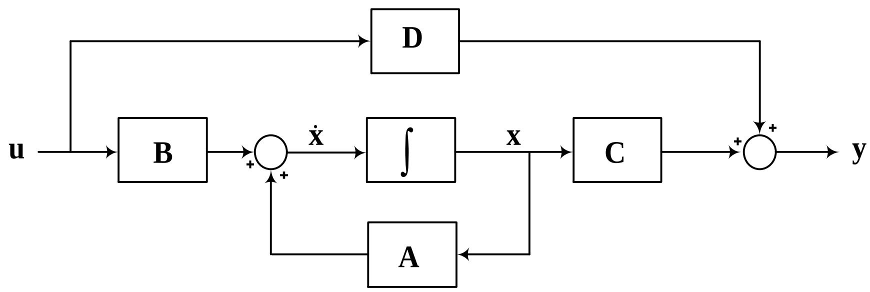


Figure (2) Block diagram for continuous-time state variable system

Ex1: Consider the inertial plant which is described by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2}$$

The equivalent differential equation is

$$\ddot{y} = u(t)$$

Now define the two required state variables as

$$x_1 = y$$

and

$$x_2 = \dot{y} = \dot{x}_1$$

so the differential equations governing the system are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u(t)$$

or in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

and the measurement (output) equation is

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Often the process of selection of system state variables in physical problems is not straightforward.

Solution of the State Equation

1. Homogeneous Equation:

Let us consider first the homogeneous form of Eq.(10), where $\mathbf{u}(t) = 0$, or

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{X}(t) \tag{12}$$

Taking the Laplace transform of this equation to yield

$$s\,\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\,\mathbf{X}(\mathbf{s}) \tag{13}$$

Rearranging gives

$$[s\,\mathbf{I} - \mathbf{A}]\,\mathbf{X}(s) = \mathbf{x}(0)$$

and solving for $\mathbf{X}(s)$, one can get

$$\mathbf{X}(s) = [s\,\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{x}(0)$$
$$= \phi(s)\,\mathbf{x}(0) \tag{14}$$

where $\phi(s) = [s\,\mathbf{I} - \mathbf{A}]^{-1}$ is called the "state transition matrix". Let us now invert the Laplace transform of Eq.(14) to yield

$$\mathbf{X}(t) = \mathcal{L}^{-1}\{ [s\,\mathbf{I} - \mathbf{A}]^{-1} \}\,\mathbf{x}(0)$$
$$= \mathcal{L}^{-1}\{ \phi(s) \}\,\mathbf{x}(0)$$

or

$$\mathbf{X}(t) = \phi(t)\,\mathbf{x}(0) \tag{15}$$

where $\phi(t)$ is given by

$$\phi(t) = \mathcal{L}^{-1}\{ [s\,\mathbf{I} - \mathbf{A}]^{-1} \} = \mathcal{L}^{-1}\,\phi(s) \tag{16}$$

2. Nonhomogeneous Equation:

Now let us consider the forced system, where $\mathbf{u}(t) \neq 0$, then

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \tag{17}$$

Let us, as before, take the Laplace transform to yield

$$s\,\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\,\mathbf{X}(\mathbf{s}) + \mathbf{B}\,\mathbf{u}(s)$$

or

$$s\,\mathbf{X}(s) - \mathbf{A}\,\mathbf{X}(\mathbf{s}) = \mathbf{x}(0) + \mathbf{B}\,\mathbf{u}(s)$$

and now solving for the s-domain solution $\mathbf{X}(s)$,

$$\mathbf{X}(s) = \left[s\,\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{x}(0) + \left[s\,\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B}\,\mathbf{u}(s) \tag{18}$$

Inverting the Laplace transform, one can get

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\left[s\,\mathbf{I} - \mathbf{A}\right]^{-1}\right\}\mathbf{x}(0) + \mathcal{L}^{-1}\left\{\left[s\,\mathbf{I} - \mathbf{A}\right]^{-1}\,\mathbf{B}\,\mathbf{u}(s)\right\}$$

or

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\phi(s)\right\}\mathbf{x}(0) + \mathcal{L}^{-1}\left\{\phi(s)\,\mathbf{B}\,\mathbf{u}(s)\right\} \tag{19}$$

☞ The first term of Eq.(19) is the same as given for the homogeneous problem, and can be rewritten in terms of state transition matrix and the state initial condition as: $\phi(t)\,\mathbf{x}(0)$.

☞ The second term is the product of two s-domain matrices $\phi(s)$ and $\mathbf{u}(s)$. It is well-known that the inverse of two functions in s-domains is equal to convolution of their corresponding time-domain functions, i.e.; if $F_1(s)$ and $F_2(s)$ are two functions in s-domain, then the following relationship is satisfied:

$$\mathcal{L}^{-1}\left\{F_1(s)\,F_2(s)\right\} = \int_0^t f_1(t-\tau)\,f_2(\tau)\,d\tau \tag{20}$$

Now letting $F_1(s) = \left[s\,\mathbf{I} - \mathbf{A}\right]^{-1} = \phi(s)$ and $F_2(s) = \mathbf{u}(s)$, the second term in Eq.(19) can be written as follows:

$$\mathcal{L}^{-1}\left\{\phi(s)\,\mathbf{B}\,\mathbf{u}(s)\right\} = \int_0^t \phi(t-\tau)\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau \tag{21}$$

and Eq.(19) becomes

$$\mathbf{x}(t) = \phi(t)\,\mathbf{x}(0) + \int_0^t \phi(t-\tau)\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau \tag{22}$$

In the above discussion the solution has been considered at some time t, given initial conditions at time $t = 0$. Now let us write the expression for an arbitrary starting time t_o, or

$$\mathbf{x}(t) = \phi\,(t - t_o)\,\mathbf{x}\,(t_o) + \int_{t_0}^{t} \phi\,(t-\tau)\,\mathbf{B}\,\mathbf{u}\,(\tau)\,d\tau \tag{23}$$

State Variable of Discrete-time System

Let us now consider the system shown in Fig.(3) where a continuous-time plant is driven by a zero-order hold and the output is sampled.

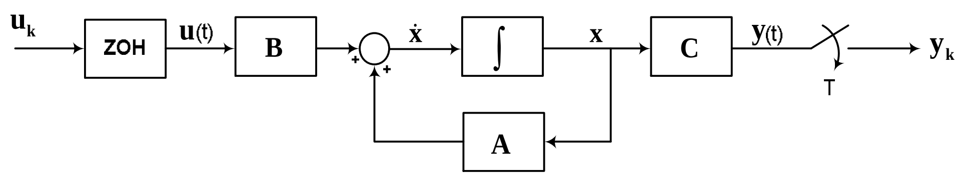


Figure (3) Continuous plant driven by a zero-order with sampled output.

If the output relation is given by

$$\mathbf{y}\,(t) = \mathbf{C}\,\mathbf{x}\,(\mathbf{t}) \qquad \{\,\mathbf{D} = 0\,\} \tag{24}$$

then the same relation must hold at the sample instants

$$\mathbf{y}\,(kT) = \mathbf{C}\,\mathbf{x}\,(kT) \tag{25}$$

so it sufficient to find the states at the sampling instants. Consider the case where $t = (kT + T)$ and $t_o = kT$, and note that the operation of zero-order hold is to create a vector $\mathbf{u}\,(kT)$ according to the relation

$$\mathbf{u}\,(t) = \mathbf{u}\,(kT) \qquad kT < t < (kT + T)$$

so from relation (22), one can get

$$\mathbf{x}\,(kT + T) = \phi\,(T)\,\mathbf{x}\,(kT) + \int_{kT}^{kT+T} \phi\,(kT + T - \tau)\,\mathbf{B}\,\mathbf{u}\,(kT)\,d\tau \tag{26}$$

Since the vector $\mathbf{u}\,(kT)$ is constant between sampling instants, it is, therefore, not a function of τ and may be extracted from the integral to the right as follows:

$$\mathbf{x}\,(kT + T) = \phi\,(T)\,\mathbf{x}\,(kT) + \left\{ \int_{kT}^{kT+T} \phi\,(kT + T - \tau)\,d\tau\,\mathbf{B} \right\} \mathbf{u}\,(kT) \tag{27}$$

Let us simplify the integral of the second term by letting $(kT + T - \tau) = \lambda$, then $d\tau = -d\lambda$, and the lower limit on λ becomes T $\{(kT + T - kT) = T\}$ and the upper limit becomes zero $\{(kT + T - (kT + T) = 0\}$. Then

$$\int_{kT}^{kT+T} \phi\,(kT+T-\tau)\; d\tau = -\int_{T}^{0} \phi\,(\lambda)\; d\lambda \tag{28}$$

or reversing the limits, one can write Eq.(27) as

$$\mathbf{x}\,(kT+T) = \phi\,(T)\;\mathbf{x}\,(kT) + \left\{\int_{0}^{T} \phi\,(\lambda)\; d\lambda\;\mathbf{B}\right\}\mathbf{u}\,(kT) \tag{29}$$

Now define the following constant matrices for constant T, or

$$\mathbf{F} = \phi\,(T) = \mathsf{L}^{-1}\left\{\left[s\,\mathbf{I} - \mathbf{A}\right]^{-1}\right\}\Big|_{t=T} \tag{30}$$

and

$$\mathbf{G} = \left\{\int_{0}^{T} \phi\,(\lambda)\; d\lambda\;\mathbf{B}\right\} \tag{31}$$

so expression (29) becomes a simple matrix-vector difference equation:

$$\mathbf{x}\,(kT+T) = \mathbf{F}\;\mathbf{x}\,(kT) + \mathbf{G}\;\mathbf{u}\,(kT) \tag{32}$$

using the shorthand notation such that $\mathbf{x}\,(kT) = \mathbf{x}\,(k)$ and $\mathbf{u}\,(kT) = \mathbf{u}\,(k)$, Eq.(32) becomes

$$\mathbf{x}\,(k+1) = \mathbf{F}\;\mathbf{x}\,(k) + \mathbf{G}\;\mathbf{u}\,(k) \tag{33}$$

with an output equation

$$\mathbf{y}\,(k) = \mathbf{C}\,\mathbf{x}\,(k) \tag{34}$$

Expressions (33) and (34) represent the discrete-time state variable of the continuous plant driven by a zero-order hold and followed by an output sampler. A block diagram for this discrete-time system is shown in Fig.(4).

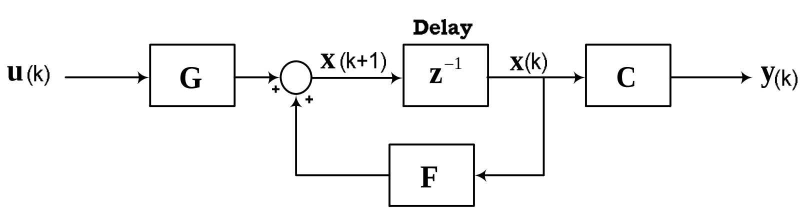


Figure (4) Block diagram for discrete-time state variable system

Ex2: For the system of Example (1), find the discrete state representation if this system is driven by a zero-order hold and followed by an output sampler.

First find the state transition matrix $\phi(t)$ using the following relation:

$$\phi(t) = \mathcal{L}^{-1}\{[s\mathbf{I} - A]^{-1}\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}$$

or

$$\phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

so the matrix $\mathbf{F}$ can be give as,

$$\mathbf{F} = \phi(T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

Now let us calculate the $\mathbf{G}$ matrix

$$\mathbf{G} = \left\{\int_0^T \phi(\lambda)\, d\lambda\, \mathbf{B}\right\}$$

$$= \left\{\int_0^T \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} d\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \begin{bmatrix} \lambda & \lambda^2/2 \\ 0 & \lambda \end{bmatrix}\Bigg|_0^T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and evaluating at the limits yields

$$\mathbf{G} = \begin{bmatrix} T & T^2/2 \\ 0 & T \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

Then the discrete state-space representation for this system is now

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

with an output relation

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Ex3: For the following continuous-time state space system,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0.5 & -0.75 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u(t)$$

Find the discrete-time state space representation.

The matrix $[s\,\mathbf{I} - \mathbf{A}]$ is

$$[s\,\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+2 & -2 \\ -0.5 & s+0.75 \end{bmatrix}$$

and the inverse matrix is

$$[s\,\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{s^2 + 2.75s + 0.5} \begin{bmatrix} s+0.75 & 2 \\ 0.5 & s+2 \end{bmatrix}$$

making the indicated partial fraction expansions after noting that the denominator roots are at $s = -0.1975$ and $s = -2.554$ gives

$$[s\,\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{0.235}{s+0.1957} + \dfrac{0.765}{s+2.554} & \dfrac{0.85}{s+0.1957} - \dfrac{0.85}{s+2.554} \\ \\ \dfrac{0.2125}{s+0.1957} - \dfrac{0.2125}{s+2.554} & \dfrac{0.765}{s+0.1957} + \dfrac{0.235}{s+2.554} \end{bmatrix}$$

Inversion of Laplace transforms yields the state transition matrix $\phi(t)$:

$$\phi(t) = \begin{bmatrix} 0.235e^{-0.1957t} + 0.765e^{-2.554t} & 0.85\,e^{-0.1957t} - 0.85\,e^{-2.554t} \\ \\ 0.2125\,e^{-0.1957t} - 0.2125\,e^{-2.554t} & 0.765\,e^{-0.1957t} + 0.235\,e^{-2.554t} \end{bmatrix}$$

If this matrix is evaluated at $t = T = 0.25$ sec, one can obtain

$$\phi(T) = \mathbf{F} = \begin{bmatrix} 0.627 & 0.361 \\ 0.0901 & 0.853 \end{bmatrix}$$

Now the matrix $\mathbf{G}$ can be obtained as follows:

$$\mathbf{G} = \left\{ \int_0^T \phi(\lambda)\, d\lambda\, \mathbf{B} \right\} = 0.5 \int_0^T \begin{bmatrix} 0.85e^{-0.1957\lambda} - 0.85e^{-2.554\lambda}) \\ \\ 0.765\,e^{-0.1957\lambda} + 0.235\,e^{-2.554\lambda} \end{bmatrix} d\lambda$$

and performing the integration and evaluating at the limits yields

$$\mathbf{G} = \begin{bmatrix} 2.17(e^{-0.1957T} - 1) - 0.166(e^{-2.554T} - 1) \\ \\ 1.95(e^{-0.1957T} - 1) + 0.046(e^{-2.554T} - 1) \end{bmatrix}$$

and evaluating at $T = 0.25\ \text{sec}$ yields

$$\mathbf{G} = \begin{bmatrix} 0.0251 \\ 0.1150 \end{bmatrix}$$

The discrete-time state equations for this system are then

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.627 & 0.361 \\ 0.0901 & 0.853 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0251 \\ 0.1150 \end{bmatrix} u(k)$$

The Matrix Exponential Series Approach:

We have seen that the state transition matrix could be evaluated by Laplace transforms as in expression (16). One may verify by differentiation that the solution to

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} \tag{35}$$

can be written as

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}t}\ \mathbf{x}(0) \tag{36}$$

$\left\{ \right.$ differentiation Eq.(36) gives $\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{e}^{\mathbf{A}t}\ \mathbf{x}(0) = \mathbf{A}\,\mathbf{x}$ which proves Eq.(35) $\left. \right\}$

where the exponential matrix is defined by

$$\mathbf{e}^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}\ t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^3 t^3 + \cdots \tag{37}$$

It is clear from comparison of relations (15) and (36) that $\mathbf{e}^{\mathbf{A}t}$ is also the state transition matrix $\phi(t)$, or

$$\phi(t) = \mathbf{e}^{\mathbf{A}t} \tag{38}$$

Since the $\mathbf{F}$ matrix in the discrete system representation is $\phi(T)$, then from Eq.(37) one can get

$$\mathbf{F} = \sum_{\mathbf{i}=0}^{\infty} \frac{\mathbf{A}^{\mathbf{i}}\,\mathbf{T}^{\mathbf{i}}}{\mathbf{i}!} \tag{39}$$

and the substitution of series expression of Eq.(37) into the integral relation (31) for the $\mathbf{G}$ matrix after integration gives, term by term,

$$\mathbf{G} = \left\{ \int_0^T \phi(\lambda)\, d\lambda\, \mathbf{B} \right\} = \left\{ \int_0^T \mathbf{e}^{\mathbf{A}\lambda}\, \mathbf{d}\lambda\, \mathbf{B} \right\}$$

$$= \int_0^T \left\{ \mathbf{I} + \mathbf{A}\,\lambda + \frac{1}{2!}\mathbf{A}^2 \lambda^2 + \frac{1}{3!}\mathbf{A}^3 \lambda^3 + \cdots \right\} d\lambda\, \mathbf{B}$$

$$= \left\{ \mathbf{I}\,\lambda + \mathbf{A}\,\frac{\lambda^2}{2!} + \mathbf{A}^2\,\frac{\lambda^3}{3!} + \cdots \right\} \Big|_0^T \;\mathbf{B}$$

$$= \left\{ \mathbf{I}\,T + \mathbf{A}\,\frac{T^2}{2!} + \mathbf{A}^2\,\frac{T^3}{3!} + \cdots \right\} \mathbf{B}$$

$$= \sum_{\mathbf{i}=0}^{\infty} \frac{\mathbf{A}^{\mathbf{i}}\, \mathbf{T}^{\mathbf{i}+1}}{(\mathbf{i}+1)!}\mathbf{B} \qquad (40)$$

If we are given a continuous-time plant in the form of A and $\mathbf{B}$ matrices and we are able to select a sampling interval T, then we may computerize the evaluation of truncated versions of relations (39) and (40) to give the discrete-time representation of the system. The matrix $\mathbf{C}$ is the same as that in the continuous-time representation.

<u>Ex</u>4: Given the inertial system of Example (1) with $\mathbf{A}$ and $\mathbf{G}$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find the $\mathbf{F}$ and $\mathbf{G}$ matrices by the method o matrix exponential series. First calculate the powers of the $\mathbf{A}$ matrix

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Similarly all higher-order powers of $\mathbf{A}$ are zero. The $\mathbf{F}$ matrix is given exactly by two terms of the series:

$$\mathbf{F} = \mathbf{I} + \mathbf{A}\,T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

Similarly $\mathbf{G}$ is given by

$$\mathbf{G} = \mathrm{T}\,\mathbf{B} + \frac{1}{2}\mathrm{T}^2\,\mathbf{A}\mathbf{B} = \begin{bmatrix} \mathrm{T}^2/2 \\ \mathrm{T} \end{bmatrix}$$

This was a very fortunate case, in that higher powers of the $\mathbf{A}$ matrix were zero. This is a seldom case in a real problem, and one will need to truncate the series and assume that enough terms are retained to give reasonable approximation of the closed form of the series.

Solved Examples

Ex1: Find the inverse z transform of

$$Y(z) = \frac{1}{(z-1)(z-0.5)}$$

Then the partial fraction expansion is

$$\frac{1}{(z-1)(z-0.5)} = A + \frac{Bz}{(z-1)} + \frac{Cz}{(z-0.5)}$$

The constant A is needed because each of the partial expansion terms has a z in the numerator. If $A \neq 0$, when we do the inverse transformation we will have a term $A\,u_1(nT)$. This is not a problem, however, since $u_1(nT)$ is a well-defined function.

One can find A by setting z=0, yielding

We see that if $Y(z)$ had a multiplicative factor z^k, $k \geq 1$, in the numerator, then the constant term would be zero. There are a number of ways to find B and C. One way is to put the partial fraction expansion over a common denominator to obtain.

Equating coefficients in the numerators on both sides of the equation yields the three linear equations

$A + B + C = 0$, $-1.5A - 0.5B - C = 0$, and $0.5A = 1$

The last equation verifies that $A = 2$. The remaining two equations then become

$B + C = -2$, $-0.5B - C = 3$

yielding $B = 2$ and $C = -4$. It is worth noting at this point that if A is not included in the partial fraction expansion, then, placing the terms over a common denominator yields

when we try to equate the numerator on both sides of the equation, we end up with

$$1 = (B + C)z^2 - (0.5B + C)z$$

which does not work. Without A we have no constant term to equate to 1.

Alternative solution:

One can write the transfer function as

$$\overline{Y}(z)=\frac{Y(z)}{z}=\frac{1}{z(z-1)(z-0.5)}=\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-0.5}$$

We now proceed just as we would for a continuous system. Thus,

$$B=(z-1)\overline{Y}(z)\Big|_{z=1}=\left[\frac{z-1}{z(z-1)(z-0.5)}\right]_{z=1}=2$$

Then,

$$\frac{Y(z)}{z}=\frac{2}{z}+\frac{2}{z-1}-\frac{4}{z-0.5}, \quad \text{and}$$

Applying the inverse z-transform to both side of this last equation then yields the same result as before, namely,

The point $z=0.5$ corresponds to the point

in the s-plane. Thus, we could write the solution as

where $\alpha=\ln\frac{0.5}{T}$

If we know T, then we can evaluate α.

Ex2: Find the inverse z transform of

$$Y(z)=\frac{0.2z}{(z-1)(z-0.6-j0.2)(z-0.6+j0.2)}$$

The multiplicative factor z in the numerator means there will be no constant term in the partial fraction expansion, which can be written as

$$\frac{Y(z)}{z}=\frac{A}{(z-1)}+\frac{B}{(z-0.6-j0.2)}+\frac{C}{(z-0.6+j0.2)}$$

The evaluation now proceeds just as it would for a Laplace transform:

The evaluation of B proceeds in the same way:

=

$$C = 1.12e^{-j2.03}$$

Then the inverse z-transform

where the damped frequency

$$\omega = \tan^{-1}\left(\frac{0.2}{0.6}\right) = 1.25$$

we can obtain $y(nT)$ as follows

with

$$\sigma = \frac{\ln\left(\sqrt{0.2^2 + 0.6^2}\right)}{T}, \ \overline{\omega} = \frac{\omega}{T} = \frac{1.25}{T}$$

<u>Ex3</u>: Solve the following difference equation:

$$y(nT + 2T) - 5y(nT + T) + 6y(nT) = u_1(nT)$$

with y(0)=1 and y(T)=0 and $u_1(nT)$ is the discrete impulse function.

Applying the z-transform to both sides:

$$[z^2Y(z) - z^2y(0) - zy(T)] - 5[zY(z) - zy(0)] + 6Y(z) = u_1(z)$$

which can be arranged as

$$Y(z) = \frac{z^2y(0) + [y(T) - 5y(0)]z}{z^2 - 5z + 6} + \frac{u_1(z)}{z^2 - 5z + 6}$$

substituting the initial conditions and since $u_1(z) = 1$ then yields

$$Y(z) = \frac{z^2 - 5z}{z^2 - 5z + 6} + \frac{1}{z^2 - 5z + 6} = \frac{z^2 - 5z + 1}{z^2 - 5z + 6}$$

we can now find $Y(z)$ by partial fraction expansion. That is, and

Finally,

and

We have to solve the expression

$$e^{\alpha T} = a$$

for α using a=2 and a=3. That is,

and

yielding

Ex4: Find the partial fraction expansion and invert the resulting transform of the following z transform function:

$$F(z) = \frac{z^2 + z}{(z-0.6)(z-0.8)(z-1)}$$

The expansion will be of the form

$$A_1 = \left.\frac{z+1}{(z-0.8)(z-1)}\right|_{z=0.6} = \frac{1.6}{(-0.2)(-0.4)} = 20$$

$$A_2 = \left.\frac{z+1}{(z-0.6)(z-1)}\right|_{z=0.8} = \frac{1.8}{(0.2)(-0.2)} = -45$$

and $$A_3 = \left.\frac{z+1}{(z-0.6)(z-0.8)}\right|_{z=1} = \frac{2}{(0.4)(0.2)} = 25$$

so upon inversion of the transform,

Ex5: Find the inverse of the following function using the method of partial fraction expansion:

$$F(z) = \frac{z^2 + z}{(z^2 - 1.13z + 0.64)(z - 0.8)} \qquad \text{(a)}$$

The chosen form will be

$$F(z) = \frac{Az^2 + Bz}{(z^2 - 1.13z + 0.64)} + \frac{Cz}{(z - 0.8)} \qquad \text{(b)}$$

First find the coefficient C

$$C = \frac{0.8 + 1}{0.64 - 1.13\,(0.8) + 0.64} = \frac{1.8}{0.376} = 4.78$$

Now find a common denominator in (b) and equate the numerator of (a) and (b) to yield the following:

Equating the coefficients of like powers of z yields

z^3: $\qquad 0 = A + 4.78$

Solving these equations for A and B yields:

$$A = -4.78$$
$$B = 2.576$$

Then the resulting z-domain function is

$$F(z) = \frac{-4.78z^2 + 2.578z}{z^2 - 1.13z + 0.64} + \frac{4.78z}{z - 0.8} = F_1(z) + \frac{4.78z}{z - 0.8} \qquad \text{(c)}$$

since

Comparing this with the first term of (c), one can obtain

$$e^{-2\alpha T} = 0.64, \qquad e^{-\alpha T} = 0.8$$

Then

$$2e^{-\alpha T}\cos(\omega T) = 1.13$$

so

$$\cos(\omega T) = 0.706$$

which implies that $\omega T = 0.786$; then

$$\sin(\omega T) = \sin(0.786) = 0.707$$

Then $F_1(z)$ can be written as

$$F_1(z) = \frac{-4.78\,(z^2 - 0.565z)}{z^2 - 1.13z + 0.64} + \frac{(2.576 - 2.7)}{0.565}\frac{0.5656z}{z^2 - 1.13z + 0.64}$$

Inverting the resulting transforms, no forgetting to add on the last term, yields a time domain sequence of

Ex6: Consider the homogeneous first-order difference equation

$$x(nT + T) - 0.8x(nT) = 0$$

with initial value $x(0) = 1$. Now take the z transform to yield

$$z\,X(z) - z\,x(0) - 0.8X(z) = 0$$

Solving for $X(z)$ yields

$$X(z) = \frac{z}{z - 0.8} = \frac{1}{1 - 0.8z^{-1}}$$

which implies that the solution is

$$x(n) = 0.8^n \qquad n = 0,1,2,\ldots$$

Ex7: Consider the same example as before with initial condition $x(0) = 1$ and an nonhomogeneuous term on the right side, or

$$x(nT + T) - 0.8\,x(nT) = 1$$

Taking the z transform yields

$$z\,X(z) - z\,x(0) - 0.8\,X(z) = \frac{z}{z - 1}$$

Solving for $X(z)$ yields

$$X(z) = \frac{2z}{z - 0.8} + \frac{z}{(z-1)(z-0.8)} = \frac{2z}{z - 0.8} + z\left[\frac{A}{z - 1} + \frac{B}{z - 0.8}\right]$$

Solving for A and B yields

So the z-domain solution

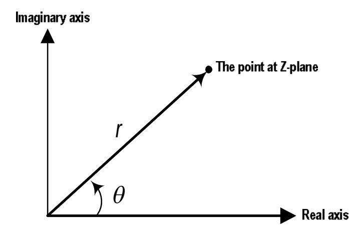
$$X(z)=\frac{2z}{z-0.8}+\frac{5z}{z-1}-\frac{5z}{z-0.8}$$

and the total solution is

$$x(n)=-3(0.8)^n+5 \qquad n=0,1,2,\ldots$$

We can check that the initial value is needed satisfied, and by substitution into the difference equation we see that it is identically satisfied

<u>Ex8:</u> Given a complex pole location in the z-plane as shown in figure below, find the damping ratio ζ , the natural frequency ω_n, and the time constant τ .



The z-plane poles occur at $z=e^{sT}$.The complex poles in s-plane appear in conjugates and have the following form

$$s_{1,2}=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Then, substitute the above expression for s into $z=e^{sT}$, we have

$$z=e^{(-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2})T} \quad z=e^{(-\zeta\omega_n)T}\ e^{\pm j\omega_n\sqrt{1-\zeta^2}}=e^{(-\zeta\omega_n)T}\ e^{\pm j\omega_n\sqrt{1-\zeta^2}T}=r\,e^{\pm\theta}$$

Hence

$$r=e^{(-\zeta\omega_n)T}$$

or

$$\zeta\,\omega_n T=-\ln(r) \qquad (1)$$

Also,

$$\omega_n\sqrt{1-\zeta^2}\;T=\theta \qquad (2)$$

Taking the ratio of Eq.(1) and (2), we obtain

$$\frac{\zeta}{\sqrt{1-\zeta^2}}=\frac{-\ln(r)}{\theta}$$

Soving this equation for ζ

$$\zeta=\frac{-\ln(r)}{\sqrt{\ln^2(r)+\theta^2}}$$

We find ω_n by substituting the last equation into Eq.(1)

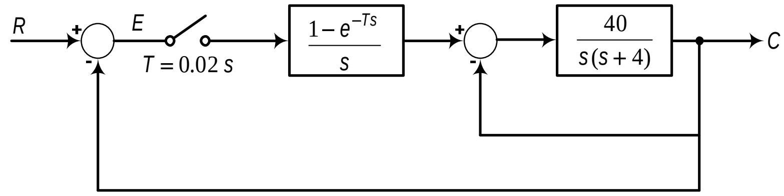
$$\omega_n = \frac{1}{T}\sqrt{\ln^2(r) + \theta^2}$$

The time constant, τ, of the pole is then given by

$$\tau = \frac{1}{\zeta\omega_n} = \frac{-T}{\ln(r)}$$

Ex9: For the closed-loop sampled-data system, find

- Closed loop pulse transfer function
- The damping ratio, the natural frequency and the time constant.
- The damping ratio, the natural frequency and the time constant for the closed-loop analogue system (with sampler and data hold removed).
- Comment the change in the above parameters with different systems.



$$G(z) = Z\left[\frac{40(1-e^{-Ts})}{s(s^2+4s+40)}\right] = \left(\frac{z-1}{z}\right) Z\left[\frac{1}{s} - \frac{s+4}{s^2+4s+40}\right]$$

$$= \left(\frac{z-1}{z}\right) Z\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+6^2} - \frac{1}{3}\frac{6}{(s+2)^2+6^2}\right]$$

With $G(z)$ in this form, we can obtain the z-transform from the tables.

with $T = 0.02\ s$, we evaluate the terms in $G(z)$,

$$2e^{-0.04}\cos(0.12) = 1.907760$$

$$\frac{1}{3}e^{-0.04}\sin(0.12) = 0.038339$$

$$e^{-0.08} = 0.923116$$

Therefore,

$$G(z) = \left(\frac{z-1}{z}\right)\left[\frac{z}{z-1} - \frac{z^2 - 0.91554\,z}{z^2 - 1.90776z + 0.92312}\right]$$

$$= \frac{0.00778z + 0.00758}{z^2 - 1.90776z + 0.92312}$$

The closed loop transfer function is then

$$\frac{C(z)}{R(z)} = \frac{0.00778z + 0.00758}{z^2 - 1.90z + 0.9307}$$

Thus, the pole locations are

$$z_{1,2} = 0.95 \pm j0.168 = 0.965e^{\pm j0.175}$$

Hence,

The closed-loop transfer function of the analogue can be expressed as

$$\frac{C(s)}{R(s)} = \frac{40}{(s+2)^2 + (8.72)^2}$$

Comparing the above to the standard second order transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\,\omega_n\,s + \omega_n^2}$$

Then,

$$\omega_n = 8.72\ rad/\sec$$

Since

$$2\zeta\omega_n = 4$$

Then

$$\zeta = \frac{2}{\omega_n} = 0.229$$

and the system time constant

$$\tau = \frac{1}{\zeta\omega_n} = 0.5\quad s$$

Thus the frequency $\omega_n = 8.72\ rad/s$ is excited by the system input. This frequency has a period of $2\pi/\omega_n$, or 0.72 s. Hence this frequency is sampled 36 times per cycle (T=0.02 s), which results in a very good description of the signal. Also, the time constant of the poles of the closed-loop transfer function, given by $1/\zeta\omega_n$ is 0.5 s. Thus, we are sampling 25 times per time constant. A rule of thumb often given for selecting sample rates is that a rate of at least five times per time constant is a good choice. Hence for this system, we would expect very little degradation in system response because of the sampling.

Ex10: Is the equation

$$u(nT) = 0.9u(nT - T) - 0.2u(nT - 2T)$$

Stable?

Taking the z transform of both side of the above equation

$$U(z) = 0.9z^{-1}U(z) - 0.2z^{-2}U(z)$$

Multiplying both side by z^2, we have

$$U(z) = \frac{1}{z^2 - 0.9z + 0.2}$$

The characteristic equation is

$$z^2 - 0.9z + 0.2 = 0$$

and the characteristic roots are $z = 0.5$ and $z = 0.5$. Since both these roots are inside the unit circle, the equation is stable.

<u>Ex11:</u> Try to represent the PID (proportional, integral and derivative controller) in a discrete form:

Continuous time PID controller can be written as

$$u(t) = K_p e(t) + K_d \dot{e}(t) + \frac{1}{K_i}\int_0^t e(t)dt = u_1(t) + u_2(t) + u_2(t)$$

where K_p, K_d and K_I are the proportional, derivative and integral gains respectively.

letting $t = nT$

$$u(nT) = u_1(nT) + u_2(nT) + u_3(nT)$$

where

$$u_1(nT) = K_p e(nT),$$

$$u_2(nT) = K_p \dot{e}(nT) = K_p \frac{[e(nT) - e(nT - T)]}{T}$$

and

Using backward approximation, the last integral of $u_3(nT)$ can be written as

$$u_3(nT) = \left[u_3(nT - T) + \frac{1}{K_i}e(nT)T\right]$$

Taking the z transform of $u_1(nT)$, $u_2(nT)$ and $u_3(nT)$, we will get

$$u_1(z) = K_p e(z)$$

$$u_2(z) = K_d \frac{e(z) - z^{-1}e(z)}{T}$$

and

$$u_3(z) - z^{-1} u_3(z) = \frac{1}{K_i} e(z)\, T \;\Rightarrow u_3(z) = \frac{1}{K_i}\left(\frac{zT}{z-1}\right) e(z)$$

Then,

$$u(z) = K_p\, e(z) + K_d \frac{z\,e(z) - e(z)}{zT} + \frac{1}{K_i}\left(\frac{zT}{z-1}\right) e(z)$$

$$z(z-1)\, u(z) = K_p\, z(z-1) e(z) + \frac{K_d}{T}(z^2 - 2z + 1) e(z) + \frac{T}{K_i}(z^2) e(z)$$

Dividing both sides by z^2

The corresponding time sequence

$$u(nT) = u(nT - T) + K_p\left[e(nT) - e(nT - T)\right] + \frac{K_d}{T}\left[e(nT) - 2e(nT - T) + e(nT - 2T)\right] + \frac{T}{K_i} e(nT)$$

or

<u>Ex12:</u> Find the transfer function $Y(z)/U(z)$ associated with the simultaneous difference equations

$$y(nT + T) - 2y(nT) + x(nT) = u(nT)$$

$$x(nT + T) - y(nT) = 3u(nT)$$

Taking the z transform of these equations while ignoring $x(0)$ and $y(0)$, we get the linear z-domain equations

$$(z-2)\, Y(z) + X(z) = U(z)$$

$$-Y(z) + z\, X(z) = 3\, U(z)$$

Now using Cramer's rule to solve for the output variable we get

$$Y(z) = \frac{\begin{vmatrix} U(z) & 1 \\ 3\,U(z) & z \end{vmatrix}}{\begin{vmatrix} z-2 & 1 \\ -1 & z \end{vmatrix}}$$

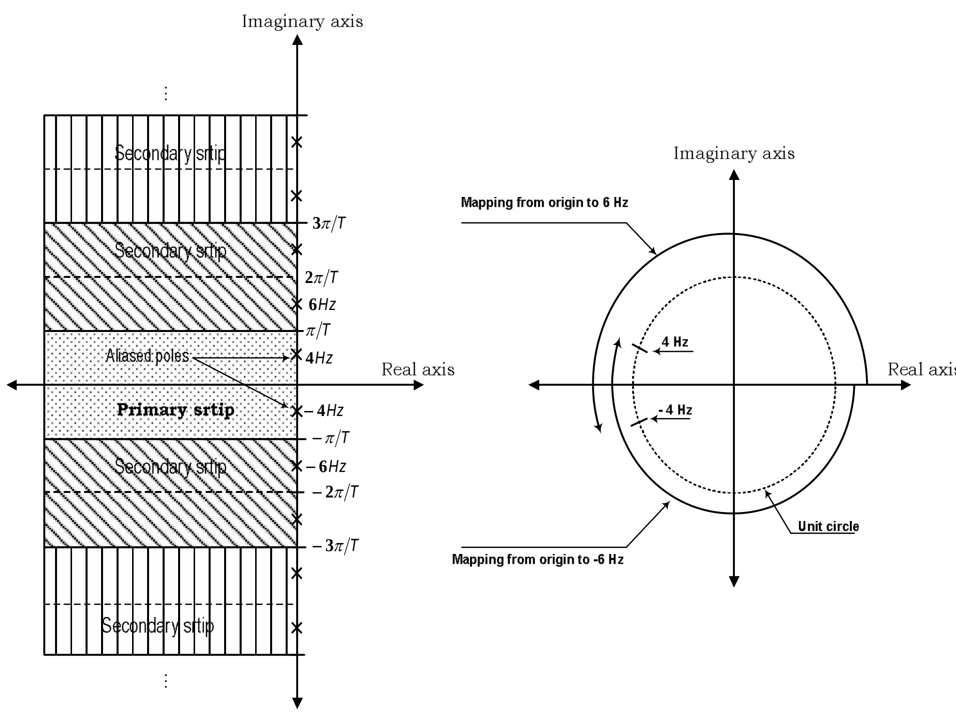
or carrying out the multiplication and combining like terms

$$\frac{Y(z)}{U(z)} = \frac{z-3}{z^2 - 2z + 1}$$

Ex13: Suppose we try to sample a 6-Hz sinwave and the sampling rate f_s is 10 Hz, so that $T = 1/f_s$ =0.1 s. Then the sine function has poles at $z = e^{\pm j6(2\pi)}$. Consider the mapping of the poles as the frequency increases from 0 to 6 Hz. The paths followed as the frequency increases are shown in Fig.(1).

Note that at a frequency of 5 Hz, the two paths meets at z=-1. The pole migrating to 6 Hz then continues on, ending up at point that corresponds to -4 Hz. The pole migrating to -6 Hz, does the same thing, ending up at a point on the unit circle that corresponds to 4 Hz. Thus, the 6 Hz sin wave will appear to be a 4 Hz sine wave.

Note that as the poles continue to migrate toward 10 Hz ($2\pi/T\ rad/s$) and -10 Hz ($-2\pi/T\ rad/s$) the aliased frequency will continue to decrease.



Ex14: Find the poles of $X^*(s)$ and $X(z)$ of $x(t) = \cos(\omega t)$.

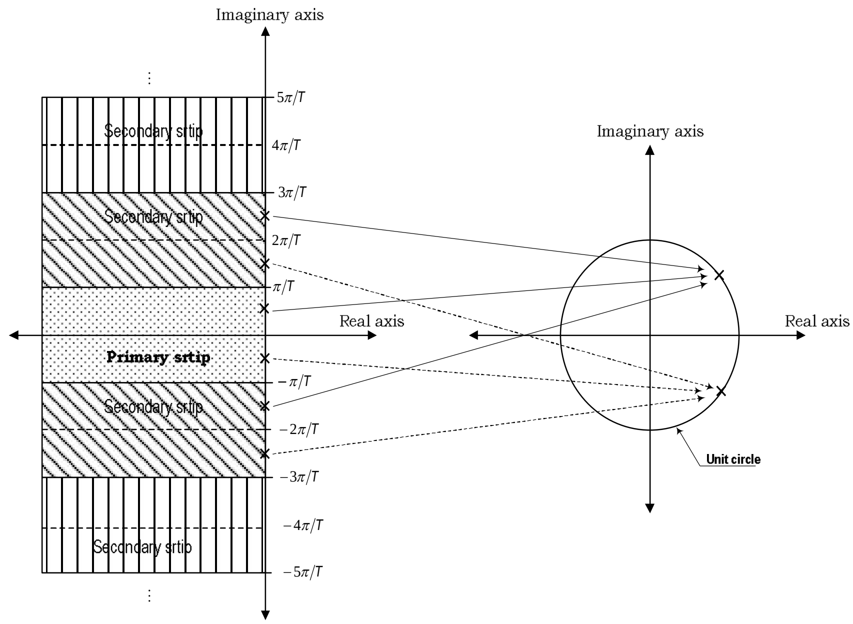
The starred transformation $X^*(s)$ is

$$X^*(s) = \sum_{n=0}^{\infty} \cos(\omega nT)e^{-nTs} = \frac{e^{sT}(e^{sT} - \cos(\omega T))}{(e^{sT} - e^{j\omega T})(e^{sT} - e^{-j\omega T})}$$

Recalling that $X^*(s)$ is periodic in s we note that $X^*(s)$ will have poles at $s = \pm j\omega \pm j\ 2n\pi/T$, $n = 1, 2, \ldots$. Thus, $X^*(s)$ has a countably infinite number of poles repeated at intervals of $2\pi/T$.

The corresponding Z transform is

The poles of $X(z)$, and the related poles of $X^*(s)$, are shown in figure below.



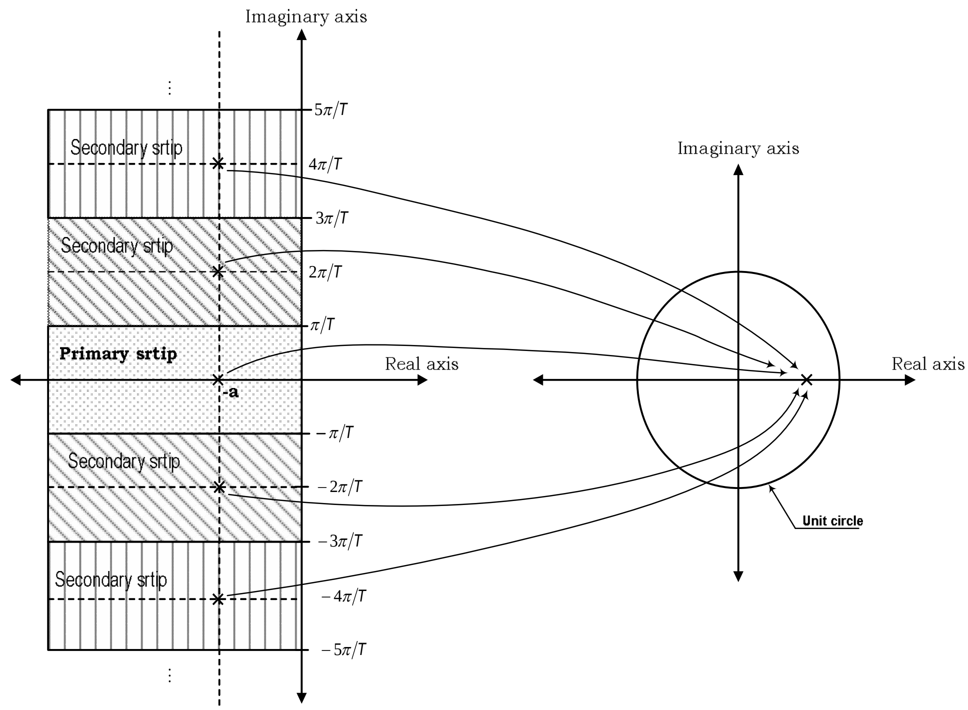
Ex15: Find the poles of $X^*(s)$ and $X(z)$ of $x(t) = e^{-at}$. The Laplace transform of this function is

$$X(s) = \frac{1}{s+a}$$

The starred transformation is

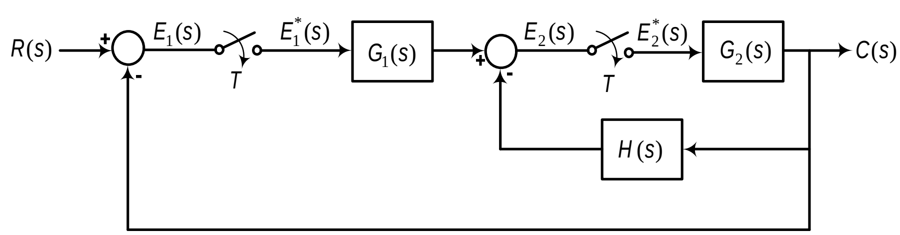
$$X^*(s) = \sum_{n=0}^{\infty} e^{-anT} e^{-nTs} = \frac{e^{sT}}{e^{sT} - e^{-aT}}$$

Recalling that $X^*(s)$ is periodic in s we note that $X^*(s)$ will have poles at $s = -a \pm j\, 2n\pi/T$, $n = 1, 2, \ldots$. Thus, $X^*(s)$ has a countably infinite number of poles, one of which is the pole of $X(s)$ and copies of this pole, repeated at intervals of $2\pi/T$. We can see from the figure below that the pole of the Laplace transform will lie in a strip of width $2\pi/T$ centered on the real axis of the s plane. This strip is called the primary strip. This pole is then repeated in the secondary strips above and below the primary strip.



- All poles of $X^*(s)$ map to the same location in the z plane, as shown in the figure. As it is known that every pole of $X(s)$ generates an infinite number of poles in $X^*(s)$.
- The mapping of $s = -a \pm j\, 2n\pi/T$, $n = 1, 2, \ldots$ of $X^*(s)$ into z plane, using $z = e^{sT}$, leads to single point at $z = e^{-aT}\, e^{j\theta}$, where $\theta = 0$.

<u>Ex16</u>: Find the pulse transfer function of the following block diagram:



The system equations are

$$E_1(s) = R(s) - G_2(s)\, E_2^*(s)$$
$$E_2(s) = G_1(s)\, E_1^*(s) - G_2(s) H(s) E_2^*(s)$$
$$C(s) = G_2(s) E_2^*(s)$$

Starring these equations gives

$$E_1^*(s) = R^*(s) - G_2^*(s)\, E_2^*(s) \qquad (1)$$

$$E_2^*(s) = G_1^*(s)\, E_1^*(s) - G_2 H^*(s) E_2^*(s) \qquad (2)$$
$$C^*(s) = G_2^*(s) E_2^*(s) \qquad (3)$$

Substitute Eq(1) into Eq.(2), and then solve for $E_2^*(s)$

$$E_2^*(s) = G_1^*(s)\left[R^*(s) - G_2^*(s) E_2^*(s)\right] - G_2 H^*(s) E_2^*(s)$$

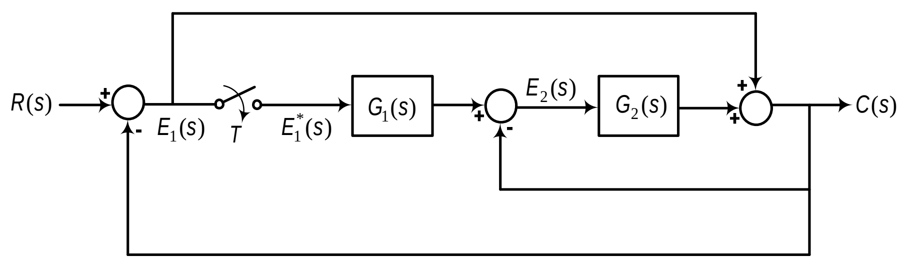
or

From Eq.(3), the sampled output $C^*(s)$ becomes

Therefore, the pulse transfer function in z transform

$$\frac{C(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1 + G_1(z)G_2(z) + G_2H(z)}$$

<u>Ex17</u>: Find the pulse transfer function of the following block diagram:



The system equations are

$$E_1(s) = R(s) - C(s) \qquad (1)$$
$$E_2(s) = G_1(s)E_1^*(s) - C(s) \qquad (2)$$
$$C(s) = G_2(s)E_2(s) + E_1(s) \qquad (3)$$

Substitute Eq. (1) and (2) into (3), one can obtain

$$C(s) = G_2(s)\left[G_1(s)E_1^*(s) - C(s)\right] + R(s) - C(s)$$
$$C(s) = G_2(s)G_1(s)E_1^*(s) - G_2(s)C(s) + R(s) - C(s) \qquad (4)$$

and solve for $C(s)$

$$C(s) = \frac{R(s)}{2 + G_2(s)} + \frac{G_1(s)G_2(s)}{2 + G_2(s)} E_1^* \qquad (5)$$

Substitute Eq.(5) into Eq.(1) to obtain

$$E_1(s) = R(s) - C(s) = R(s) - \left[\frac{R(s)}{2+G_2(s)} + \frac{G_1(s)G_2(s)}{2+G_2(s)} E_1^*(s) \right]$$

$$E_1(s) = \frac{\left[1+G_2(s)\right]R(s)}{2+G_2(s)} - \frac{G_1(s)G_2(s)}{2+G_2(s)} E_1^*(s) \tag{6}$$

Starring of Eq.(5) and Eq.(6), we obtain

$$C^*(s) = \left[\frac{R(s)}{2+G_2(s)} \right]^* + \left[\frac{G_1(s)G_2(s)}{2+G_2(s)} E_1^* \right]^*$$

$$C^*(s) = \frac{R^*(s)}{2+G_2^*(s)} + \frac{G_1G_2^*(s)}{2+G_2^*(s)} E_1^*$$

$$E_1^*(s) = \left[\frac{\left(1+G_2(s)\right)R(s)}{2+G_2(s)} \right]^* - \left[\frac{G_1(s)G_2(s)}{2+G_2(s)} E_1^*(s) \right]^*$$

$$E_1^*(s) = \frac{\left(R^*(s) + RG_2^*(s)\right)}{2+G_2^*(s)} - \frac{G_1G_2^*(s)}{2+G_2^*(s)} E_1^*(s)$$

Solving for $E_1^*(s)$

$$E_1^*(s) = \frac{\left(R^*(s) + RG_2^*(s)\right)}{\left(2+G_2^*(s) + G_1G_2^*(s)\right)}$$

Substitute the last equation into $C^*(s)$ equation

$$C^*(s) = \frac{R^*(s)}{2+G_2^*(s)} + \frac{G_1G_2^*(s)}{2+G_2^*(s)} \left[\frac{\left(R^*(s) + RG_2^*(s)\right)}{\left(2+G_2^*(s) + G_1G_2^*(s)\right)} \right]$$

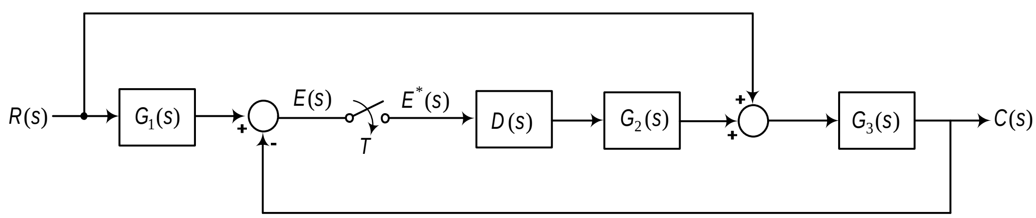
$$C^*(s) = \left[\frac{R^*(s)\left(2+G_2^*(s) + 2G_1G_2^*(s)\right) \; + G_1G_2^*(s) \; RG_2^*(s)}{\left(2+G_2^*(s) + G_1G_2^*(s)\right)\left(2+G_2^*(s)\right)} \right]$$

The corresponding z transform is

$$C(z) = \left[\frac{R(z)\left(2+G_2(z) + 2G_1G_2(z)\right) \; + G_1G_2(z) \; RG_2(z)}{\left(2+G_2(z) + G_1G_2(z)\right)\left(2+G_2(z)\right)} \right]$$

Note that no pulse transfer function is possible for this system, since the input is fed into a continuous element in the system without first being sampled.

Ex18: Find the pulse transfer function of the following block diagram:



The system equations are

$$C(s) = G_3(s)R(s) + D(s)G_2(s)G_3(s)E^*(s) \qquad (1)$$

$$E(s) = R(s)G_1(s) - C(s)$$

$$E(s) = R(s)G_1(s) - R(s)G_3(s) - D(s)G_2(s)G_3(s)E^*(s) \qquad (2)$$

Starring of both Eq.(1) and (2)

$$C^*(s) = RG_3^*(s) + DG_2\,G_3^*(s)E^*(s) \qquad (3)$$

$$E^*(s) = RG_1^*(s) - RG_3^*(s) - DG_2\,G_3^*(s)E^*(s) \qquad (4)$$

Solving for $E^*(s)$

$$E^*(s) = \frac{RG_1^*(s) - RG_3^*(s)}{1 + DG_2\,G_3^*(s)} \qquad (5)$$

Substitute the last equation into Eq.(3), we obtain

$$C^*(s) = RG_3^*(s) + DG_2\,G_3^*(s)\left(\frac{RG_1^*(s) - RG_3^*(s)}{1 + DG_2\,G_3^*(s)}\right)$$

$$C^*(s) = \left(\frac{RG_3^*(s) + DG_2\,G_3^*(s)RG_1^*(s)}{1 + DG_2\,G_3^*(s)}\right)$$

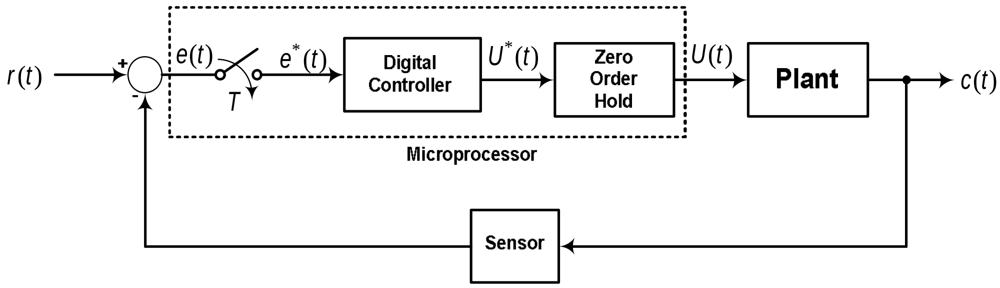
in z transform

$$C(z) = \left(\frac{RG_3(z) + DG_2G_3(z)RG_1(z)}{1 + DG_2G_3(z)} \right)$$

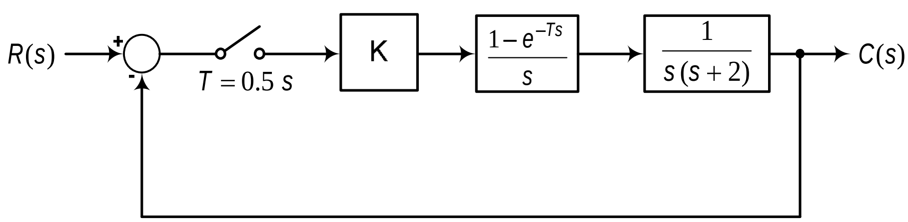
Note that no transfer function may be derived for this system.

<u>Ex19</u>: A digital control may be represented by the block diagram of figure below:

- ❑ The open-loop pulse transfer function.
- ❑ The closed-loop pulse transfer function.
- ❑ The difference equation for the discrete time response.
- ❑ Sketch the unit step response assuming zero initial conditions.
- ❑ The steady-state value of the system output.



The above digital control system may be redrawn as follows:



❑

$$G(s) = K\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s\ (s+2)}\right)$$

given K=1

$$G(s) = \left(1-e^{-Ts}\right)\left\{\frac{1}{s^2\,(s+2)}\right\}$$

Partial fraction expansion

or

$$1 = s(s+2)A + (s+2)B + s$$

Equating coefficients gives

$$A = -0.25$$
$$B = 0.5$$
$$C = 0.25$$

Then, G(s) becomes

$$G(s) = \left(1 - e^{-Ts}\right)\left\{\frac{-0.25}{s} + \frac{0.5}{s^2} + \frac{0.25}{(s+2)}\right\}$$

or

$$G(s) = 0.25\left(1 - e^{-Ts}\right)\left\{-\frac{1}{s} + \frac{2}{s^2} + \frac{1}{(s+2)}\right\}$$

Taking z-transforms

$$G(z) = 0.25(1 - z^{-1})\left\{-\frac{z}{z-1} + \frac{2Tz}{(z-1)^2} + \frac{z}{(z - e^{-2T})}\right\}$$

Given T=0.5 seconds

$$G(z) = 0.25\left(\frac{z-1}{z}\right) z \left\{-\frac{1}{z-1} + \frac{2 \times 0.5}{(z-1)^2} + \frac{1}{(z-0.368)}\right\}$$

Hence

which simplifies to give the open-loop transfer function

$$G(z) = \left(\frac{0.092 + 0.066}{z^2 - 1.368z + 0.368}\right)$$

- The close-loop pulse transfer function is

which simplifies to give the closed-loop pulse transfer function

$$\frac{C(z)}{R(z)} = \left(\frac{0.092z + 0.066}{z^2 - 1.276z + 0.434}\right)$$

or

$$\frac{C(z)}{R(z)} = \frac{0.092z^{-1} + 0.066z^{-2}}{1 - 1.276z^{-1} + 0.434z^{-2}}$$

- The last transfer function can be expressed as a difference equation

- Using the final value theorem, one can obtain

$$c(\infty) = \underbrace{\lim}_{z \to 1} \left[\left(\frac{z-1}{z} \right) \frac{C(z)}{R(z)} \times R(z) \right]$$

$$c(\infty) = \underbrace{\lim}_{z \to 1} \left[\left(\frac{z-1}{z} \right) \left\{ \frac{0.092z + 0.066}{z^2 - 1.276z + 0.434} \right\} \times \frac{z}{z-1} \right]$$

$$c(\infty) = \left(\frac{0.092 + 0.066}{1 - 1.276 + 0.434} \right) = 1$$

Hence there is no steady-state error.

- The response in the following figure is constructed solely from the knowledge of the two previous sampled outputs and the two previously sampled inputs.