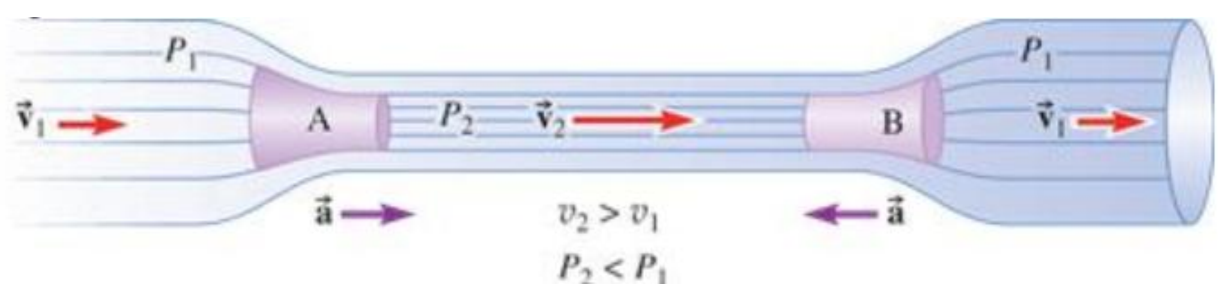
Lecture Seven

Bernoulli Equation

7.1 Introduction

The continuity equation relates the flow velocities of an ideal fluid at two different points, based on the change in cross-sectional area of the pipe. According to the continuity equation, the fluid must speed up as it enters a constriction (below figure) and then slow down to its original speed when it leaves the constriction. Using energy ideas, we will show that the pressure of the fluid in the constriction (P2) cannot be the same as the pressure before or after the constriction (P1). For horizontal flow the speed is higher where the pressure is lower. This principle is often called the Bernoulli effect.



Bernoulli's equation is a fundamental principle in fluid mechanics that describes the conservation of energy in a flowing fluid. It states that for an incompressible, frictionless flow, the total mechanical energy of the fluid remains constant.

Mathematically, it is expressed as:

$$P + \frac{1}{2}\rho V^2 + \rho g h = \text{constant}$$

Bernoulli's equation relates the pressure, flow speed, and height at two points in an ideal fluid. Although we derived Bernoulli's equation in a relatively simple situation, it applies to the flow of any ideal fluid as long as points 1 and 2 are on the same streamline.

where:

- P = Pressure (Pa)
- ρ = Fluid density (kg/m^3)
- V = Velocity of the fluid (m/s)
- g = Acceleration due to gravity (9.81 m/s^2)
- h = Height above a reference level (m)

The units of Bernoulli's equations are J/m^3 .The Bernoulli equation along the stream-line is a statement of the work energy theorem. As the particle moves, the pressure and gravitational forces can do work, resulting in a change in the kinetic energy.

Bernoulli's equation is valid under the following assumptions:

- ✅ Steady flow (does not change with time)
- ✅ Incompressible fluid (constant density)
- ✅ No viscosity (ideal fluid, no energy loss due to friction)
- ✅ Flow along a streamline

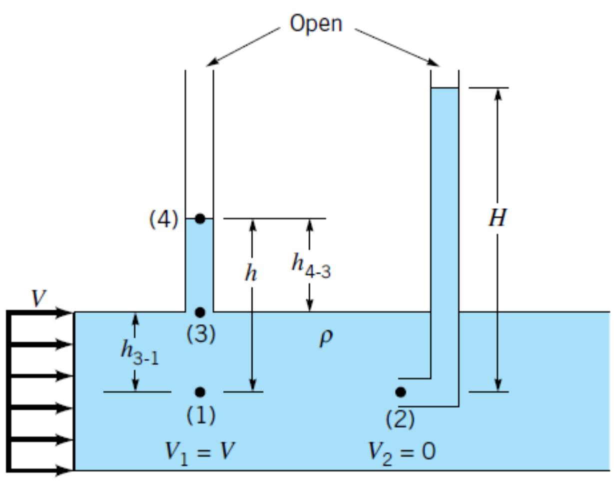
7.2 Dynamic, static and hydrostatic pressures

Static pressure is the pressure as measured moving with the fluid. (e.g. static with fluid). This is the p term in Bernoulli's equation. Imagine moving along the fluid with a pressure gauge.

Sometimes the term $\rho g h$ in Bernoulli's equation is called the hydrostatic pressure. (e.g. it is the change in pressure due to change in elevation.)

Dynamic pressure is a pressure that occurs when kinetic energy of the flowing fluid is converted into pressure rise. This is the pressure associated with the $\frac{1}{2}\rho V^2$ term in Bernoulli's equation.

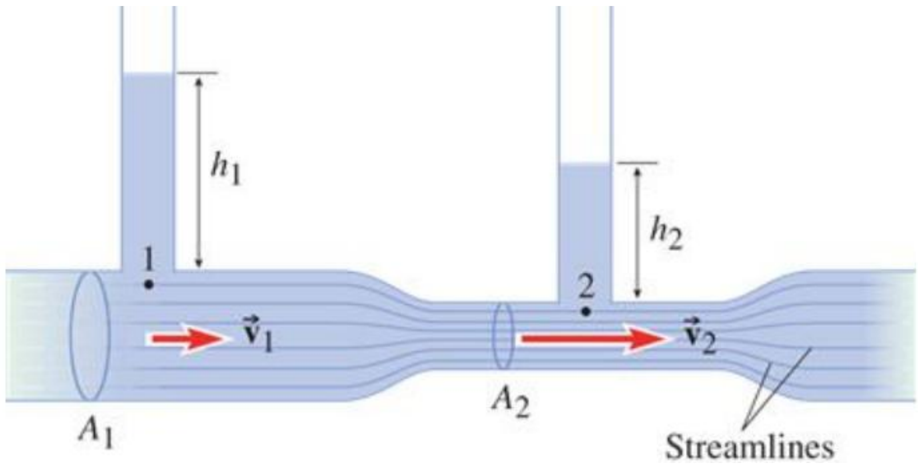
Example 1



$$
\begin{aligned}
p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2 &= p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 \\
p_2 + 0 + \rho g z_1 &= p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 \quad (v_2 = 0) \\
p_2 &= p_1 + \frac{1}{2}\rho v_1^2
\end{aligned}
$$

***Example* 2**

Suppose that the pipe in question carries water, $A_1 = 2\ A_2$, and the fluid heights in the vertical tubes are $h_1 = 1.20$ m and $h_2 = 0.80$ m. (a) Find the ratio of the flow speeds V_2/V_1. (b) Find the gauge pressures P_1 and P_2. (c) Find the flow speed V_1 in the pipe.



Sol

Neither of the two flow speeds is given. We need more than Bernoulli's equation to solve this problem. Since we know the ratio of the areas, the continuity equation gives us the ratio of the speeds. The height of the water in the vertical tubes enables us to find the pressures at points 1 and 2. The fluid pressure at the bottom of each vertical tube is the same as the pressure of the moving fluid just beneath each tube—otherwise, water would flow into or out of the vertical tubes until the pressure equalized. The water in the vertical tubes is static, so the gauge pressure at the bottom is $P = \rho gh$. Once we have the ratio of the speeds and the pressures, we apply Bernoulli's equation.

(a) From the continuity equation, the product of flow speed and area must be the same at points 1 and 2. Therefore,

$$\frac{v_2}{v_1} = \frac{A_1}{A_2} = 2.0$$

The water flows twice as fast in the constriction as in the rest of the pipe.

(b) The gauge pressures are:

$$P_1 = \rho g h_1 = 1000 \text{ kg/m}^3 \times 9.80 \text{ N/kg} \times 1.20 \text{ m} = 11.8 \text{ kPa}$$
$$P_2 = \rho g h_2 = 1000 \text{ kg/m}^3 \times 9.80 \text{ N/kg} \times 0.80 \text{ m} = 7.8 \text{ kPa}$$

(c) Now we apply Bernoulli's equation. We can use gauge pressures as long as we do so on both sides:

$$P_1 + \rho g y_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \rho g y_2 + \tfrac{1}{2}\rho v_2^2$$

Since the tube is horizontal, y1 = y2 and we can ignore the small change in gravitational potential energy density ρgh. Then:

$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2$$

We are trying to find V_1, so we can eliminate V_2 by substituting $V_2 = 2\ V_1$:

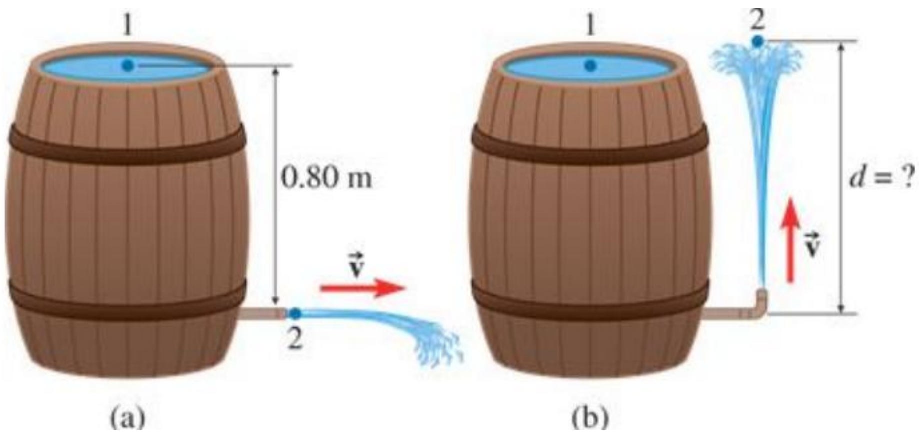
$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho (2.0 v_1)^2$$

$$P_1 - P_2 = 1.5 \rho v_1^2$$

$$v_1 = \sqrt{\frac{11\,800 \text{ Pa} - 7800 \text{ Pa}}{1.5 \times 1000 \text{ kg/m}^3}} = 1.6 \text{ m/s}$$

***Example* 3**

A barrel full of rainwater has a spigot near the bottom, at a depth of 0.80 m beneath the water surface. (a) When the spigot is directed horizontally and is opened, how fast does the water come out? (b) If the opening points upward, how high does the resulting "fountain" go?



Sol

(a)

$$P_1 + \rho g y_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \rho g y_2 + \tfrac{1}{2}\rho v_2^2$$

Since $P_1 = P_2$, Bernoulli's equation is:

$$\rho g y_1 + \tfrac{1}{2}\rho v_1^2 = \rho g y_2 + \tfrac{1}{2}\rho v_2^2$$

Point 1 is 0.80 m above point 2, so:

$$y_1 - y_2 = 0.80 \text{ m}$$

The speed of the emerging water is V_2. What is V_1, the speed of the water at the surface? The water at the surface is moving slowly, since the barrel is draining. The continuity equation requires that:

$$v_1 A_1 = v_2 A_2$$

Since the cross-sectional area of the spigot A_2 is much smaller than the area of the top of the barrel A_1, the speed of the water at the surface V_1 is negligibly small compared with V_2. Setting $V_1 = 0$, Bernoulli's equation reduces to:

$$\rho g y_1 = \rho g y_2 + \tfrac{1}{2}\rho v_2^2$$

After dividing through by ρ, we solve for V_2:

$$g(y_1 - y_2) = \tfrac{1}{2}v_2^2$$

$$v_2 = \sqrt{2g(y_1 - y_2)} = 4.0 \text{ m/s}$$

(b) Now take point 2 to be at the top of the fountain. Then $V_2 = 0$ and Bernoulli's equation reduces to:

$$\rho g y_1 = \rho g y_2$$

The "fountain" goes right back up to the top of the water in the barrel!!!
In reality, the fountain does not reach as high as the original water level; some energy is dissipated due to viscosity and air resistance.