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Lecture 3

“Laplace Inverse Using Partial Fraction”

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CHAPTER 3

INVERSION BY PARTIAL FRACTIONS

Our study of the application of Laplace transforms to linear differential equations with constant coefficients has enabled us to rapidly establish the Laplace transform of the solution. We now wish to develop methods for inverting the transforms to obtain the solution in the time domain. In the first part of this chapter we give a series of examples that illustrate the partial fraction technique. After a generalization of these techniques, we proceed to a discussion of the qualitative information that can be obtained from the transform of the solution without inverting it.

The equations to be solved are all of the general form

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = r(t)$$

The unknown function of time is $x(t)$, and $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants. The given function $r(t)$ is called the *forcing function.* In addition, for all problems of interest in control system analysis, the initial conditions are given. In other words, values of $x, dx/dt, \ldots, d^{n-1}x/dt^{n-1}$ are specified at time 0. The problem is to determine $x(t)$ for all $t \geq 0$.

3.1 PARTIAL FRACTIONS

In the series of examples that follow, the technique of partial fraction inversion for solution of this class of differential equations is presented.

Example 3.1. Solve

$$\frac{dx}{dt} + x = 1$$
$$x(0) = 0$$

Application of the Laplace transform yields

$$sx(s) + x(s) = \frac{1}{s}$$

or

$$x(s) = \frac{1}{s(s+1)}$$

The theory of partial fractions enables us to write this as

$$x(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \tag{3.1}$$

where A and B are constants. Hence, from Table 2.1, it follows that

$$x(t) = A + Be^{-t} \tag{3.2}$$

Therefore, if A and B were known, we would have the solution. The conditions on A and B are that they must be chosen to make Eq. (3.1) an identity in s.

To determine A, multiply both sides of Eq. (3.1) by s.

$$\frac{1}{s+1} = A + \frac{Bs}{s+1} \tag{3.3}$$

Since this must hold for all s, it must hold for $s = 0$. Putting $s = 0$ in Eq. (3.3) yields

$$A = 1$$

To find B, multiply both sides of Eq. (3.1) by $s + 1$.

$$\frac{1}{s} = \frac{A}{s}(s+1) + B \tag{3.4}$$

Since this must hold for all s, it must hold for $s = -1$. This yields

$$B = -1$$

This procedure for determining the coefficients is called the *Heaviside expansion.* There is an easy way to visualize the Heaviside procedure and quickly determine the coefficients of the partial fraction expansion (A and B in this case). Considering Eq. (3.1), we can determine A, the numerator of the $1/s$ factor, by ignoring (or "covering up") this term in the denominator of $x(s)$ and letting all the remaining s's equal the value of s that makes the "covered up" term equal to zero. The other coefficients are found in a similar manner.

For example, to solve for A, we "cover up" the s factor and let all the other s values equal 0.

$$A = \frac{1}{\cancel{s}(\underbrace{\cancel{s}}_{0} + 1)} = 1$$

Similarly for B, we cover up the $s + 1$ term and let the other s values equal -1, so

$$B = \frac{1}{\underbrace{\cancel{s}}_{-1}\ \cancel{(s + 1)}} = \frac{1}{-1} = -1$$

Cross-multiplication (as well as the quick visualization method) works for distinct roots (non-repeated factors in the denominator) and in a limited way for repeated roots. We will discuss the case of repeated roots shortly.

Now that we've found A and B, we have

$$x(s) = \frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{s + 1} \tag{3.5}$$

and therefore,

$$x(t) = 1 - e^{-t} \tag{3.6}$$

Equation (3.5) may be checked by putting the right side over a common denominator, and Eq. (3.6) by substitution into the original differential equation and initial condition.

Example 3.2. Chemical mixing scenario revisited. In Chap. 2 we solved the chemical mixing scenario problem to the point where we had obtained the transformed solution to the material and energy balances. The transformed solutions, Eqs. (2.10) and (2.11), are repeated in this example for convenience.

$$C_a(s) = \frac{2}{s(5s + 1)} + \frac{15}{5s + 1} \tag{2.10}$$

We can now invert this expression for the concentration in the tank to the time domain.

Considering the first term on the left-hand side, we can separate it into partial fractions by using the same method that was employed in Example 3.1.

$$\frac{2}{s(5s + 1)} = \frac{\frac{2}{5}}{s\left(s + \frac{1}{5}\right)} = \frac{A}{s} + \frac{B}{s + \frac{1}{5}} = \frac{2}{s} + \frac{-2}{s + \frac{1}{5}}$$

Equation (2.10) may now be written as

$$C_a(s) = \frac{2}{s(5s+1)} + \frac{15}{5s+1} = \frac{2}{s} - \frac{2}{s+\frac{1}{5}} + \frac{3}{s+\frac{1}{5}} = \frac{2}{s} + \frac{1}{s+\frac{1}{5}}$$

We can now readily invert this expression to the time domain

$$C_a(t) = 2 + e^{-t/5}$$

This is the same solution that we previously obtained by separation and integration of the original mass balance differential equation in the time domain, which is plotted in Fig. 2–3.

Similarly, we can obtain the time domain solution for the temperature in the mixing vessel by inverting Eq. (2.11).

$$T(s) = \frac{70/s + 5(80)}{5s+1} = \frac{70}{s(5s+1)} + \frac{400}{5s+1} \tag{2.11}$$

Separating the right-hand side by using partial fractions, we get

$$T(s) = \frac{70}{s(5s+1)} + \frac{400}{5s+1} = \frac{70}{s} + \frac{-70}{s+\frac{1}{5}} + \frac{80}{s+\frac{1}{5}} = \frac{70}{s} + \frac{10}{s+\frac{1}{5}}$$

$$T(t) = 70 + 10e^{-t/5}$$

This is the same solution that we previously obtained by separation and integration of the original energy balance differential equation in the time domain which is plotted in Fig. 2–7.

Example 3.3. Solve

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 4 + e^{2t}$$

$$x(0) = 1 \quad x'(0) = 0 \quad x''(0) = -1$$

Taking the Laplace transform of both sides yields

$$\left[s^3x(s) - s^2 + 1\right] + 2\left[s^2x(s) - s\right] - [sx(s) - 1] - 2x(s) = \frac{4}{s} + \frac{1}{s-2}$$

Solving algebraically for $x(s)$, we find

$$x(s) = \frac{s^4 - 6s^2 + 9s - 8}{s(s-2)(s^3 + 2s^2 - s - 2)} \tag{3.7}$$

The cubic in the denominator may be factored, and $x(s)$ is expanded in partial fractions.

$$x(s) = \frac{s^4 - 6s^2 + 9s - 8}{s(s-2)(s+1)(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+1} + \frac{D}{s+2} + \frac{E}{s-1} \tag{3.8}$$

To find A, multiply both sides of Eq. (3.8) by s and then set $s = 0$; the result is

$$A = \frac{-8}{(-2)(1)(2)(-1)} = -2$$

The other constants are determined in the same way. The procedure and results are summarized in the following table.

To determine	**Multiply Eq. (3.8) by**	**and set *s* to**	**Result**
B	$s-2$	2	$B = 1/12$
C	$s+1$	-1	$C = 11/3$
D	$s+2$	-2	$D = -17/12$
E	$s-1$	1	$E = 2/3$

Accordingly, the solution to the problem is

$$x(t) = -2 + \tfrac{1}{12}e^{2t} + \tfrac{11}{3}e^{-t} - \tfrac{17}{12}e^{-2t} + \tfrac{2}{3}e^{t}$$

A comparison between this method and the classical method, as applied to Example 3.2, may be profitable. In the classical method for solution of differential equations, we first write down the characteristic function of the homogeneous equation:

$$s^3 + 2s^2 - s - 2 = 0$$

This must be factored, as was also required in the Laplace transform method, to obtain the roots -1, -2, and $+1$. Thus, the complementary solution is

$$x_c(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{t}$$

Furthermore, by inspection of the forcing function, we know that the particular solution has the form

$$x_p(t) = A + Be^{2t}$$

The constants A and B are determined by substitution into the differential equation and, as expected, are found to be -2 and $\frac{1}{12}$, respectively. Then

$$x(t) = -2 + \tfrac{1}{12}e^{2t} + C_1e^{-t} + C_2e^{-2t} + C_3e^{t}$$

Using MATLAB for Symbolic Processing—Partial Fractions

Remember that we have previously declared some variables symbolic (a, k, x, y, z, t, and s). MATLAB does not have a built-in function for performing partial fractions. However, we can force MATLAB to do the work for us by taking advantage of two other MATLAB functions, **`diff`** and **`int`.** We have MATLAB integrate $x(s)$, which it does internally by using partial fractions, and then immediately differentiate the resulting expression. The result will be the partial fraction expansion of $x(s)$.

Let's have MATLAB find the partial fraction expansion represented by Eq. (3.8).

```
x=(s^4-6*s^2+9*s-8)/s/(s-2)/(s+1)/(s+2)/(s-1)

x=
(s^4-6*s^2+9*s-8)/s/(s-2)/(s+1)/(s+2)/(s-1)

diff(int(x))

ans=
-2/s+1/12/(s-2)+11/3/(s+1)-17/12/(s+2)+2/3/(s-1)
pretty(ans)

                          1              1     17   1                1
      -2 / s + 1 / 12 ------- + 11 / 3 ------- - -- ------- + 2 / 3 -------
                        s - 2          s + 1     12 s + 2           s - 1
```

Thus, MATLAB arrives at the same result as we did by hand.

and constants C_1, C_2, and C_3 are determined by the three initial conditions. The Laplace transform method has systematized the evaluation of these constants, avoiding the solution of three simultaneous equations. Four points are worth noting:

1. In both methods, one must find the roots of the characteristic equation. The roots give rise to terms in the solution *whose form is independent of the forcing function.* These terms make up the *complementary solution.*

2. The forcing function gives rise to terms in the solution *whose form depends on the form of the forcing function and is independent of the left side of the equation.* These terms comprise the *particular solution.*

3. The only interaction between these sets of terms, i.e., between the right side and left side of the differential equation, occurs in the evaluation of the constants involved.

4. The only effect of the initial conditions is in the evaluation of the constants. This is so because the initial conditions affect only the numerator of $x(s)$, as may be seen from the solution of this example.

In the three examples we have discussed, the denominator of $x(s)$ factored into real factors only. In the next example, we consider the complications that arise when the denominator of $x(s)$ has complex factors.

Using MATLAB for Symbolic Processing—Solving ODEs

MATLAB can symbolically solve ODEs. It uses the DSOLVE command for this purpose. We illustrate the use of this command with Example 3.2.

The problem consisted of

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 4 + e^{2t}$$
$$x(0) = 1 \quad x'(0) = 0 \quad x''(0) = -1$$

The **DSOLVE** command is straighforward for solving this equation.

```
dsolve('D3x+2*D2x-Dx-2*x=4+exp(2*t)','x(0)=1','Dx(0)=0','D2x(0)=-1')
```

```
ans=
-1/12*exp(2*t)*(24*exp(-2*t)-1)+2/3*exp(t)-17/12*exp(-2*t)+11/3*exp(-t)
```

`expand(ans)` This command multiplies out the expression to make it easier to compare with our original answer.

```
ans=
-2+1/12*exp(t)^2+2/3*exp(t)-17/12/exp(t)^2+11/3/exp(t)
```

which is the same result we obtained by hand: $x(t) = -2 + \frac{1}{12}e^{2t} + \frac{11}{3}e^{-t} - \frac{17}{12}e^{-2t} + \frac{2}{3}e^{t}$

We can verify this result with MATLAB by inverting the partial fraction expansion we obtained with MATLAB previously.

```
ilaplace(-2/s+1/12/(s-2)+11/3/(s+1)-17/12/(s+2)+2/3/(s-1))
```

```
ans=
-2+1/12*exp(2*t)+11/3*exp(-t)-17/12*exp(-2*t)+2/3*exp(t)
```

The result is the same!

Example 3.4. Inversion of a transform that has complex roots in the denominator. Solve

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 2$$
$$x(0) = 0 \qquad x'(0) = 0$$

Application of the Laplace transform yields

$$x(s) = \frac{2}{s\left(s^2 + 2s + 2\right)}$$

The quadratic term in the denominator may be factored by use of the quadratic formula. The roots are found to be $-1 - j$ and $-1 + j$. If we use these complex roots in the partial fraction expansion, the algebra can get quite tedious. We present a method to obtain the partial fraction expansion for the case of complex roots, without resorting to the use of complex algebra.

Avoiding the use of complex algebra with a quadratic term. If we choose not to factor the quadratic term, we can use an alternate form of the partial fraction expansion.

$$x(s) = \frac{2}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

Note that the second term of the expansion has the unfactored quadratic in the denominator. The numerator of each term in the expansion is a polynomial in s of one less degree than the denominator, hence the $Bs + C$ in the numerator (a first-order numerator with a second-order denominator). As before, we can determine A.

$$A = \frac{2}{0 + 2(0) + 2} = 1$$

So we now have

$$x(s) = \frac{2}{s(s^2 + 2s + 2)} = \frac{1}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

Clearing the denominator on the left-hand side, we obtain

$$2 = s^2 + 2s + 2 + Bs^2 + Cs$$

Collecting like terms, we get

$$(B + 1)s^2 + (2 + C)s + 2 = 2$$

We now match coefficients of like terms on the left and right sides of the equation.

$$\begin{array}{lll} s^2: & B + 1 = 0 & B = -1 \\ s: & 2 + C = 0 & C = -2 \end{array}$$

Thus,

$$x(s) = \frac{1}{s} - \frac{s + 2}{s^2 + 2s + 2} \tag{3.9}$$

To invert the second term, we complete the square in the denominator to get a familiar transform. Remember that for a perfect square, the quadratic must have the form $s^2 + \alpha s + (\alpha/2)^2 = (s + \alpha/2)^2$, where the constant is one-half the middle coefficient squared. The second term on the right-hand side becomes

$$\frac{s+2}{s^2+2s+2} = \frac{s+2}{(s^2+2s+1)+2-1} = \frac{s+2}{(s+1)^2+1}$$

where we've added and subtracted one-half the middle coefficent squared, $(2/2)^2 = 1$, so the denominator remains unchanged. The transform of the solution is now

$$x(s) = \frac{1}{s} - \frac{s+2}{(s+1)^2+1} \tag{3.10}$$

One last modification of the second term is required before inversion. A term of this type will lead to a sine term and a cosine term in the solution. From Table 2.1, we see that

$$L\left\{e^{-at}\sin(kt)\right\} = \frac{k}{(s+a)^2+k^2} \tag{3.11a}$$

$$L\left\{e^{-at}\cos(kt)\right\} = \frac{s+a}{(s+a)^2+k^2} \tag{3.11b}$$

Note that everywhere s appears in these forms, it appears as the quantity $s + a$. Thus, comparing these transforms with Eq. (3.10), we see that we need an $s + 1$ term in the numerator, to go with the $s + 1$ in the denominator. So we regroup as

$$x(s) = \frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1^2} = \frac{1}{s} - \frac{s+1}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}$$

We can easily invert these terms to obtain the solution to the differential equation.

$$x(t) = 1 - e^{-t}(\cos t + \sin t)$$

We now summarize the steps in this method for inverting quadratic terms with complex roots while avoiding the use of complex algebra.

Step 1. Form the partial expansion term for the quadratic with a first-order term in s in the numerator.
Step 2. Determine the numerators of the other terms in the expansion, using the Heaviside expansion.
Step 3. Cross-multiply the equation for $x(s)$ by the denominator of $x(s)$, and equate coefficients of like terms to determine the constants in the numerator of the quadratic term.
Step 4. Complete the square for the quadratic term.
Step 5. Regroup the terms in the numerator, such that if the quadratic is now $(s + a)^2$, everywhere else that s appears, it appears as $s + a$.
Step 6. Invert the resulting two terms to a sine and a cosine term (probably multiplied by an exponential).

In the next example, an exceptional case is considered; the denominator of $x(s)$ has *repeated roots.* The procedure in this case will vary slightly from that of the previous cases.

Example 3.5. Inversion of a transform with repeated roots. Solve

$$\frac{d^3x}{dt^3} + \frac{3d^2x}{dt^2} + \frac{3dx}{dt} + x = 1$$
$$x(0) = x'(0) = x''(0) = 0$$

Application of the Laplace transform yields

$$x(s) = \frac{1}{s\left(s^3 + 3s^2 + 3s + 1\right)}$$

Factoring and expanding in partial fractions, we find

$$x(s) = \frac{1}{s(s+1)^3} = \frac{A}{s} + \frac{B}{(s+1)^3} + \frac{C}{(s+1)^2} + \frac{D}{s+1} \tag{3.12}$$

As in the previous cases, to determine A, multiply both sides by s and then set s to zero. This yields

$$A = 1$$

Multiplication of both sides of Eq. (3.12) by $(s+1)^3$ results in

$$\frac{1}{s} = \frac{A(s+1)^3}{s} + B + C(s+1) + D(s+1)^2 \tag{3.13}$$

Setting $s = -1$ in Eq. (3.13) gives

$$B = -1$$

Having found A and B, we introduce these values into Eq. (3.12) and place the right side of the equation over a common denominator; the result is

$$\frac{1}{s(s+1)^3} = \frac{(s+1)^3 - s + Cs(s+1) + Ds(s+1)^2}{s(s+1)^3} \tag{3.14}$$

Expanding the numerator of the right side gives

$$\frac{1}{s(s+1)^3} = \frac{(1+D)s^3 + (3+C+2D)s^2 + (2+C+D)s + 1}{s(s+1)^3} \tag{3.15}$$

We now equate the numerators on each side to get

$$1 = (1+D)s^3 + (3+C+2D)s^2 + (2+C+D)s + 1$$

Equating the coefficients of like powers of s gives

$$\begin{aligned} 1 + D &= 0 \\ 3 + C + 2D &= 0 \\ 2 + C + D &= 0 \end{aligned}$$

Solving these equations gives $C = -1$ and $D = -1$.

The final result is then

$$x(s) = \frac{1}{s} - \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} - \frac{1}{s+1} \tag{3.16}$$

By referring to Table 2.1, this can be inverted to

$$x(t) = 1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right) \tag{3.17}$$

The reader should verify that Eq. (3.16) placed over a common denominator results in the original form

$$x(s) = \frac{1}{s(s+1)^3}$$

and that Eq. (3.17) satisfies the differential equation and initial conditions.

The result of Example 3.5 may be generalized. The appearance of the factor $(s + a)^n$ in the denominator of $x(s)$ leads to n terms in the partial fraction expansion

$$\frac{C_1}{(s+a)^n}, \ \frac{C_2}{(s+a)^{n-1}}, \ \ldots, \ \frac{C_n}{s+a}$$

The constant C_1 can be determined as usual by multiplying the expansion by $(s + a)^n$ and setting $s = -a$. The other constants are determined by the method shown in Example 3.5. These terms, according to Table 2.1, lead to the following expression as the inverse transform:

$$\left[\frac{C_1}{(n-1)!}t^{n-1} + \frac{C_2}{(n-2)!}t^{n-2} + \cdots + C_{n-1}t + C_n\right]e^{-at} \tag{3.18}$$

It is interesting to recall that in the classical method for solving these equations, one treats repeated roots of the characteristic equation by postulating the form of Eq. (3.18) and selecting the constants to fit the initial conditions.

[redacted]

3A.8 GENERAL DISCUSSION OF PARTIAL FRACTIONS ON A QUADRATIC TERM

In Chap. 3 we discussed how to express a quadratic term in the denominator of a fraction using partial fractions without resorting to complex algebra if we had complex roots. For completeness, we present the method using complex algebra.

Consider the general expression involving a quadratic term

$$x(s) = \frac{F(s)}{s^2 + \alpha s + \beta} \tag{3A.3}$$

where $F(s)$ is some function of s (say, $1/s$). Expanding the terms on the right side gives

$$x(s) = F_1(s) + \frac{Bs + C}{s^2 + \alpha s + \beta} \tag{3A.4}$$

where $F_1(s)$ represents other terms in the partial fraction expansion. First solve for B and C algebraically by placing the right side over a common denominator and equating the coefficients of like powers of s. The next step is to express the quadratic term in the form

$$s^2 + \alpha s + \beta = (s + a)^2 + k^2$$

The terms a and k can be found by solving for the roots of $s^2 + \alpha s + \beta = 0$ by the quadratic formula to give $s_1 = -a + jk$ and $s_2 = -a - jk$. The quadratic term can now be written

$$s^2 + \alpha s + \beta = (s - s_1)(s - s_2) = (s + a - jk)(s + a + jk) = (s + a)^2 + k^2$$

Equation (3A.4) now becomes

$$x(s) = F_1(s) + \frac{Bs + C}{(s + a)^2 + k^2} \tag{3A.5}$$

The numerator of the quadratic term is written to correspond to the transform pairs given by Eqs. (3.11*a*) and (3.11*b*).

$$Bs + C = B\left(s + a + \frac{C/B - a}{k}k\right) = B(s + a) + \frac{C - aB}{k}k$$

Equation (3A.5) becomes

$$x(s) = F_1(s) + B\frac{s + a}{(s + a)^2 + k^2} + \frac{C - aB}{k}\frac{k}{(s + a)^2 + k^2}$$

Applying the transform pairs of Eqs. (3.11*a*) and (3.11*b*) to the quadratic terms on the right gives

$$\boxed{x(t) = F_1(t) + Be^{-at}\cos kt + \left(\frac{C - aB}{k}\right)e^{-at}\sin kt} \tag{3A.6}$$

where $F_1(t)$ is the result of inverting $F_1(s)$; B and C are coefficients of polynomial $Bs + C$ in numerator of quadratic term; and a and k correspond to the roots of the quadratic, roots $= -a \pm kj$. Let's use this generalized approach to solve a problem that we're already familiar with, Example 3.4.

Recall from Eq. (3.9) that

$$x(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2} = \frac{1}{s} + \frac{-s - 2}{s^2 + 2s + 2}$$

The roots of the quadratic in the denominator are

$$\text{Roots} = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm j$$

Summarizing the constants required for Eq. (3A.6), we have

A	B	C	a	k	$F_1(s)$
1	−1	−2	1	1	$1/s$

Substituting these quantities into Eq. (3A.6), we obtain the solution

$$x(t) = 1 + (-1)e^{-t}\cos t + \frac{-2 - (1)(-1)}{1}e^{-t}\sin t = 1 - e^{-t}(\cos t + \sin t)$$

which is the same as our previous result.

We now apply this method to another example.

Example 3A.9. Solve

$$x(s) = \frac{1}{s\left(s^2 - 2s + 5\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 5}$$

Applying the quadratic equation to the quadratic term gives

$$\text{Roots} = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2j$$

Thus, we find that $a = -1$ and $k = 2$. Solving for A, B, and C gives $A = \frac{1}{5}$, $B = -\frac{1}{5}$, and $C = \frac{2}{5}$. Introducing these values into the expression for $x(s)$ and applying Eq. (3A.6) give

$$x(t) = \tfrac{1}{5} - \tfrac{1}{5}e^{t}\cos 2t + \tfrac{1}{10}e^{t}\sin 2t$$

3A.9 USING COMPLEX ALGEBRA FOR A QUADRATIC TERM

Reworking Example 3.4 using the complex roots of the quadratic, we can arrive at the partial fraction expansion

$$x(s) = \frac{2}{s(s+1+j)(s+1-j)} = \frac{A}{s} + \frac{B}{s+1+j} + \frac{C}{s+1-j} \quad (3A.7)$$

where A, B, and C are constants to be evaluated, so that this relation is an identity in s. The presence of complex factors does not alter the Heaviside procedure at all. However, the computations are more tedious.

To obtain A, multiply Eq. (3A.7) by s and set $s = 0$:

$$A = \frac{2}{(1+j)(1-j)} = 1$$

To obtain B, multiply Eq. (3A.7) by $s + 1 + j$ and set $s = -1 - j$:

$$B = \frac{2}{(-1-j)(-2j)} = \frac{-1-j}{2}$$

To obtain C, multiply Eq. (3A.7) by $s + 1 - j$ and set $s = -1 + j$:

$$C = \frac{2}{(-1+j)(2j)} = \frac{-1+j}{2}$$

Therefore,

$$x(s) = \frac{1}{s} + \frac{-1-j}{2}\frac{1}{s+1+j} + \frac{-1+j}{2}\frac{1}{s+1-j}$$

This is the desired result. To invert $x(s)$, we may now use the fact that $1/(s + a)$ is the transform of e^{-t}. The fact that a is complex does not invalidate this result, as can be seen by returning to the derivation of the transform of e^{-at}. The result is

$$x(t) = 1 + \frac{-1-j}{2}e^{-(1+j)t} + \frac{-1+j}{2}e^{-(1-j)t}$$

By using the identity

$$e^{(a+jb)t} = e^{at}(\cos bt + j\sin bt)$$

this can be converted to

$$x(t) = 1 - e^{-t}(\cos t + \sin t)$$

The details of this conversion are recommended as an exercise for the reader.

A more general discussion of this case will promote understanding. It was seen in Example 3.4 that the complex conjugate roots of the denominator of $x(s)$ gave rise to a pair of complex terms in the partial fraction expansion. The constants in these terms, B

and C, proved to be complex conjugates $(-1-j)/2$ and $(-1+j)/2$. When these terms were combined through a trigonometric identity, it was found that the complex terms canceled, leaving a real result for $x(t)$. Of course, it is necessary that $x(t)$ be real, since the original differential equation and initial conditions are real.

This information may be utilized as follows: The general case of complex conjugate roots arises in the form

$$x(s) = \frac{F(s)}{(s + k_1 + jk_2)(s + k_1 - jk_2)} \tag{3A.8}$$

where $F(s)$ is some real function of s.

For instance, in Example 3A.9 we had

$$F(s) = \frac{2}{s} \qquad k_1 = 1 \qquad k_2 = 1$$

Expanding Eq. (3A.8) in partial fractions gives

$$\frac{F(s)}{(s + k_1 + jk_2)(s + k_1 - jk_2)} = F_1(s) + \left(\frac{a_1 + jb_1}{s + k_1 + jk_2} + \frac{a_2 + jb_2}{s + k_1 - jk_2}\right) \tag{3A.9}$$

where a_1, a_2, b_1, and b_2 are the constants to be evaluated in the partial fraction expansion and $F_1(s)$ is a series of fractions arising from $F(s)$.

Again, in Example 3A.9,

$$a_1 = -\frac{1}{2} \qquad a_2 = -\frac{1}{2} \qquad b_1 = -\frac{1}{2} \qquad b_2 = \frac{1}{2} \qquad F_1(s) = \frac{1}{s}$$

Now, since the left side of Eq. (3A.9) is real for all real s, the right side must also be real for all real s. Since two complex numbers will add to form a real number if they are complex conjugates, the right side will be real *for all real s* if and only if the two terms are complex conjugates. Since the denominators of the terms are conjugates, this means that the numerators must also be conjugates, or

$$a_2 = a_1$$
$$b_2 = -b_1$$

This is exactly the result obtained in the specific case of Example 3.4. With this information, Eq. (3A.9) becomes

$$\frac{F(s)}{(s + k_1 + jk_2)(s + k_1 - jk_2)} = F_1(s) + \left(\frac{a_1 + jb_1}{s + k_1 + jk_2} + \frac{a_1 - jb_1}{s + k_1 - jk_2}\right) \tag{3A.10}$$

Hence, it has been established that terms in the inverse transform arising from the complex conjugate roots may be written in the form

$$(a_1 + jb_1)e^{(-k_1 - jk_2)t} + (a_1 - jb_1)e^{(-k_1 + jk_2)t}$$

Again, by using the identity

$$e^{(C_1+jC_2)t} = e^{C_1t}\left(\cos C_2t + j \sin C_2t\right)$$

this reduces to

$$2e^{-k_1t}\left(a_1 \cos k_2t + b_1 \sin k_2t\right) \tag{3A.11}$$

Let us now rework Example 3A.9, using Eq. (3A.11). We return to the point at which we arrived, by our usual techniques, with the conclusion that

$$B = \frac{-1 - j}{2}$$

Comparison of Eqs. (3A.7) and (3A.10) and the result for B show that we have two possible ways to assign a_1, b_1, k_1, and k_2 so that we match the form of Eq. (3A.10). They are

$$\begin{array}{ll} a_1 = -\frac{1}{2} & a_1 = -\frac{1}{2} \\ b_1 = -\frac{1}{2} & b_1 = \frac{1}{2} \end{array}$$

or

$$\begin{array}{ll} k_1 = 1 & k_1 = 1 \\ k_2 = 1 & k_2 = -1 \end{array}$$

The first way corresponds to matching the term involving B with the first term of the conjugates of Eq. (3A.10), and the second to matching it with the second term. *In either case,* substitution of these constants into Eq. (3A.11) yields

$$-e^{-t}(\cos t + \sin t)$$

which is, as we have discovered, the correct term in $x(t)$.

What this means is that one can proceed directly from the evaluation of one of the partial fraction constants, in this case B, to the complete term in the inverse transform, in this case $-e^{-t}$ (cos t + sin t). It is not necessary to perform all the algebra, since it has been done in the general case to arrive at Eq. (3A.11).

Another example will serve to emphasize the application of this technique.

Example 3A.10. Solve

$$\frac{d^2x}{dt^2} + 4x = 2e^{-1} \qquad x(0) = x'(0) = 0$$

The Laplace transform method yields

$$x(s) = \frac{2}{\left(s^2 + 4\right)(s + 1)}$$

Factoring and expanding into partial fractions give

$$\frac{2}{(s+1)(s+2j)(s-2j)} = \frac{A}{s+1} + \frac{B}{s+2j} + \frac{C}{s-2j} \tag{3A.12}$$

Multiplying Eq. (3A.12) by $s+1$ and setting $s=-1$ yields

$$A = \frac{2}{(-1+2j)(-1-2j)} = \frac{2}{5}$$

Multiplying Eq. (3A.12) by $s+2j$ and setting $s=-2j$ yields

$$B = \frac{2}{(-2j+1)(-4j)} = \frac{-2+j}{10}$$

Matching the term

$$\frac{(-2+j)/10}{s+2j}$$

with the first term of the conjugates of Eq. (3A.10) requires that

$$a_1 = -\frac{2}{10} = -\frac{1}{5} \qquad b_1 = \frac{1}{10} \qquad k_1 = 0 \qquad k_2 = 2$$

Substituting in Eq. (3A.11) results in

$$-\tfrac{2}{5}\cos 2t + \tfrac{1}{5}\sin 2t$$

Hence the complete answer is

$$x(t) = \tfrac{2}{5}e^{-t} - \tfrac{2}{5}\cos 2t + \tfrac{1}{5}\sin 2t$$

Readers should verify that this answer satisfies the differential equation and initial conditions. In addition, they should show that it can also be obtained by matching the term with the second term of the conjugates of Eq. (3A.10) or by determining C instead of B.

SUMMARY

In this appendix, we have presented and discussed several properties of Laplace transforms. As we continue our studies with first-order systems, these properties will prove quite useful in our understanding and analysis of the process dynamics.

PROBLEMS

3A.1. If a forcing function $f(t)$ has the Laplace transform

$$f(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s}$$

graph the function $f(t)$.

The appendix to this chapter is a grouping of several Laplace transform theorems that will find later application. In addition, a discussion of the impulse function $\delta(t)$ is presented there. Unavoidably, this appendix is rather dry. It may be desirable for the reader to skip directly to Chap. 4, where our control studies begin. At each point where a theorem of App. 3A is applied, reference to the appropriate section of the appendix can be made.

PROBLEMS

3.1. Solve the following by using Laplace transforms.

(*a*) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x = 1 \qquad x(0) = x'(0) = 0$

(*b*) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 1 \qquad x(0) = x'(0) = 0$

(*c*) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + x = 1 \qquad x(0) = x'(0) = 0$

Sketch the behavior of these solutions on a single graph. What is the effect of the coefficient of dx/dt?

3.2. Solve the following differential equations by Laplace transforms.

(*a*) $\dfrac{d^4x}{dt^4} + \dfrac{d^3x}{dt^3} = \cos t \qquad x(0) = x'(0) = x'''(0) = 0 \qquad x''(0) = 1$

(*b*) $\dfrac{d^2q}{dt^2} + \dfrac{dq}{dt} = t^2 + 2t \qquad q(0) = 4 \qquad q'(0) = -2$

3.3. Invert the following transforms.

(*a*) $\dfrac{3s}{\left(s^2 + 1\right)\left(s^2 + 4\right)}$

(*b*) $\dfrac{1}{s\left(s^2 - 2s + 5\right)}$

(*c*) $\dfrac{3s^3 - s^2 - 3s + 2}{s^2(s - 1)^2}$

3.4. Expand the following functions by partial fraction expansion. Do *not* evaluate coefficients or invert expressions.

(*a*) $X(s) = \dfrac{2}{(s + 1)\left(s^2 + 1\right)^2(s + 3)}$

(*b*) $X(s) = \dfrac{1}{s^3(s + 1)(s + 2)(s + 3)^3}$

(*c*) $X(s) = \dfrac{1}{(s + 1)(s + 2)(s + 3)(s + 4)}$

3.5. (*a*) Invert: $x(s) = 1/[s(s + 1)(0.5s + 1)]$
(*b*) Solve: $dx/dt + 2x = 2 \qquad x(0) = 0$

3.6. Obtain $y(t)$ for

(*a*) $y(s) = \dfrac{s + 1}{s^2 + 2s + 5}$

(*b*) $y(s) = \dfrac{s^2 + 2s}{s^4}$

(*c*) $y(s) = \dfrac{2s}{(s - 1)^3}$

3.7. (*a*) Invert the following function.

$$y(s) = 1/\left(s^2 + 1\right)^2$$

(*b*) Plot y versus t from 0 to 3π.

3.8. Determine $f(t)$ for $f(s) = 1/[s^2 (s + 1)]$.

3.9. Solve the following differential equations.

(*a*) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = u(t) \qquad x(0) = x'(0) = 0$

(*b*) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = u(t) \qquad x(0) = x'(0) = 1$

(*c*) $2\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = u(t) \qquad x(0) = x'(0) = 0$

3.10. Use the trigonometric identities below to express the solution to Prob. 3.9*c* in terms of sine only. (*Note:* A sine and a cosine wave with the same frequency can be expressed as a single sine wave of the same frequency. The resulting sine wave will have a different amplitude and be phase-shifted from the original waves. This result will be important when we discuss frequency response in Chap. 15.)

$$a_1 \cos b + a_2 \sin b = a_3 \sin(b + \Phi)$$
$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

3.11. Find $f(t)$ if $F(s)$ is

(*a*) $\dfrac{1}{(s + 1)^3(s + 2)}$

(*b*) $\dfrac{s + 1}{s^2 + 2s + 5}$

(*c*) $\dfrac{s^2 - s - 6}{s^3 - 2s^2 - s + 2}$

(*d*) $\dfrac{s + 1}{s^2(s + 2)}$

(*e*) $\dfrac{1}{s(As+1)(Bs+1)}$

(*f*) $\dfrac{s+1}{s(2s+1)}$

(*g*) $\dfrac{s+1}{s^2+3s+1}$

(*h*) $\dfrac{s+1}{s^2(2s+1)}$

3.12. Find the solution to the following set of equations.

$$\left.\begin{aligned}\frac{dx_1}{dt} &= 2x_1 + 3x_2 + 1\\ \frac{dx_2}{dt} &= 2x_1 + x_2 + e^t\end{aligned}\right\} \qquad x_1(0) = x_2(0) = 0$$

Hint: Transformed equations can be manipulated algebraically to solve for each unknown (i.e., two equations in two unknowns) and then inverted separately.

3.13. Use MATLAB.

(*a*) Find the partial fraction expansions for Prob. 3.11.

(*b*) Invert the transforms in Prob. 3.11, using the **`ILAPLACE`** command.

(*c*) Graph the solutions to Prob. 3.11 (skip Prob. 3.11*e*).

3.14. Use the MATLAB **`DSOLVE`** command to solve Prob. 3.12.

3.15. Use the MATLAB **`DSOLVE`** command to solve Prob. 3.9.

3.16. (*a*) Solve the differential equations in Prob. 2.3, using partial fractions.

(*b*) Use the MATLAB **`DSOLVE`** command to solve the ODEs in Prob. 2.3.