Example:-

Find the response of the mass - spring – damper system under a unit impulse (Dirac delta function) at time $t = 1$. Let m = 1 kg , c = 3 and k = 2

Solution

The equation of motion in this case becomes $y''(t) + 3y'(t) + 2y(t) = \delta(t-1)$ Note that initial conditions = 0

Taking L.T of both sides of differential equation

$$(s^2 + 3s + 2)Y(s) = e^{-s} \quad \Rightarrow \quad Y(s) = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{k_1}{s+1} + \frac{k_2}{s+2}\right)e^{-s}$$

$$Y(s) = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}$$

By corollary 2 the $y(t) = \text{L.T}^{-1}Y(s)$ is

$$y(t) = e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)$$

$$y(t) = \begin{cases} 0 & 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & if \quad t > 1 \end{cases}$$

Example:- What is y(t) if $Y(s) = \frac{s+2}{(s^2+4s+5)^2}$

Solution

$$s^2 + 4s + 5 + 4 - 4 = s^2 + 4s + 4 + 1 = (s+2)^2 + 1$$

so $Y(s)$ can be written as $Y(s) = \frac{s+2}{[(s+2)^2+1]^2}$

$$\text{LT}^{-1} \text{ of } \{\text{Y}(s)\} = e^{-2t}\,\text{LT}^{-1}\frac{s}{[s^2+1]^2}$$

$$\frac{s}{[s^2+1]^2} = [s^2+1]^{-2}\ s$$

then from integration theorem $\left\{\frac{f(t)}{t}\right\} = \text{LT}^{-1} \text{ of } \int_s^\infty \phi(s)\,ds$

let $\phi(s) = [s^2+1]^{-2}\ s \cdot \frac{2}{2}$, $\int_s^\infty \phi(s)\,ds = \frac{1}{2}\int_s^\infty [s^2+1]^{-2}\ 2s\,ds = -\frac{1}{2}\frac{1}{s^2+1}\Big|_s^\infty$

$$\int_s^\infty \phi(s)\,ds = \frac{1}{2}\frac{1}{s^2+1} \qquad f(t) = \frac{1}{2}\,t\,\sin t$$

then $y(t) = \frac{1}{2}\,t\,e^{-2t}\sin t$

Problems

Find the L.T of each of the following functions:

1- $u(t-a)$

2- $u(t - e^{-t})$

3- $t^2 u(t-2)$

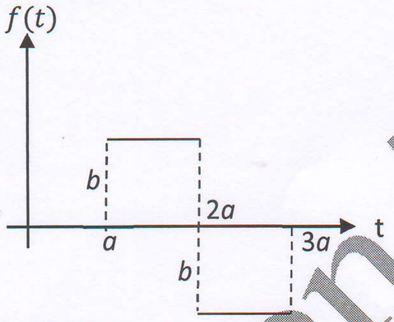
4- $\cos t\, u(t-1)$

5- $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t \end{cases}$

6- $(t) = \begin{cases} t & 0 < t < 2 \\ 2 & 2 < t \end{cases}$

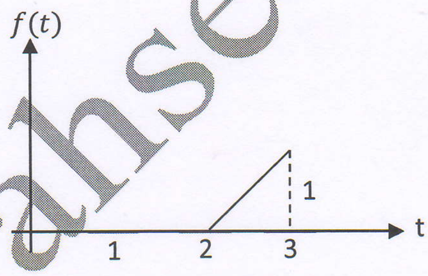
7- The function graphed in Figure (1)

Figure (1)



8- The function graphed in Figure (2)

Figure (2)



Power Series :-

We shall examine sequences and series where the n-th term is a function of $u_n(x)$. For example $u_0(x) = 1$, $u_1(x) = x$, $u_2(x) = x^2, \ldots .. u_n(x) = x^n$, then the sequences $\{u_n(x)\} = \{x^n\}$ converges to zero, if $-1 < x < 1$; converges to 1 if $x = 1$; and diverges elsewhere. When we connect these terms with plus signs, we get the :

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

Which it is converges to $\frac{1}{1-x}$ if $|x| < 1$ and diverges otherwise. We say that $\sum x^n$ defines a function on (-1 ,1), whether we known a closed formula for $\sum x^n$ or not.

Formal Power Series :-

If we have the following series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots,$$

This called formal power series

a_n = are constant (independent of x)

x = is a variable whose domain at the moment may be any set of real number.

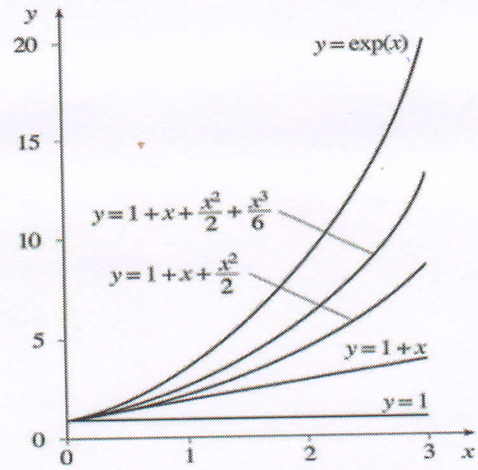
Taylor Polynomials :-

Every function f that is defined in a neighborhood of $x = 0$ and has finite derivatives $f', f'', \ldots . f^{(n)}$ at $= 0$, generate polynomials $p_0(x), p_1(x), p_2(x), \ldots, p_n(x)$ that approximate the given function $f(x)$ for value of x near $x = 0$. The approximations usually get better as the degree of the polynomial increases.

Let the polynomial p_k be

$p_k(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ ----------------------------------(*)

Where $a_0, a_1, a_2, \ldots a_k$ are coefficients. We focus our attention on apportion of the curve $y = f(x)$, which intersection at point $A(0, f(0))$, as shown in fig.



We must find the constants $a_0, a_1, a_2, \dots, a_k$ to approximate $f(x)$, as follows:-

Put $k = 4$

$\therefore\ p_4(x) = a_0 + a_1 x + a_2x^2 + a_3x^3 + a_4x^4$ ------------------------------ (1)

$p_4'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$ ------------------------------ (2)

$p_4''(x) = \ \ +2a_2 + 6a_3x + 12a_4x^2$ ------------------------------ (3)

$p_4'''(x) = \qquad +6a_3 + 24a_4x$ ------------------------------ (4)

$p''''_{(x)} = \qquad\qquad +24\, a_4$ ------------------------------ (5)

Putting $x = 0$,

$$p_4(0) = f(0),$$
$$p_4'(0) = f'(0)$$
$$p_4''(0) = f''(0)$$
$$p_4'''(0) = f'''(0)$$
$$p_4''''(0) = f''''(0)$$

And solving the above equations for $a_0, a_1, a_2, a_3, a_4,$ we obtain,

$$a_0 = f(0), a_1 = f'(0), a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}, a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}, a_4 = \frac{f''''(0)}{24} = \frac{f''''}{4!} \dots\dots$$

$$, a_k = \frac{f^{(k)}(0)}{k!}$$

Thus , sub. Above values into eq. (*)

$$p_k(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots \frac{f^k(0)}{k!}x^k$$

Which it is called the k-th order, Taylor Polynomial generated by f at $x = 0$

Example:- Find the Taylor polynomials $p_n(x)$ generated by $f(x) = e^x$ at $x = 0$

Solution:- The given function and its derivative are.

$f(x) = e^x, f'(x) = e^x, \dots, f^{(n)}(x) = e^x$

So $f(0) = e^x = 1$,

$f'(0) = 1$,

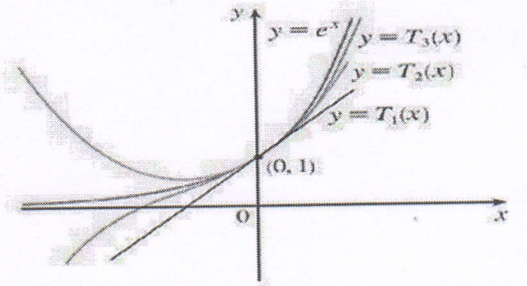
,

,

,

$f^{(n)}(0) = 1$

$p_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$



Maclaurin Series :-

If $k \longrightarrow \infty$ The Taylor Polynimial will be ⇒ Maclaurian Series

⇒ The Maclaurian Series generated by f

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \text{ -------------- } (7)$$

Example:- Find the Maclaurian Series generated by e^x at $x = 0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{n=\infty} \frac{x^n}{n!}$$

Example:- Find the Maclaurian Series generated by $\cos x$ at $x = 0$.

Solution:-

$f(x) = \cos x \quad , f'(x) = -\sin x$

$f''(x) = -\cos x , f'''(x) = \sin x$

$f''''(x) = \cos x \ , \ f^{(5)}(x) = -\sin x$

,

,

$f^{2n}(x) = (-1)^n \cos x \ , \ f^{(2n+1)}(x) = (-1)^{n+1} \sin x$

Putting $x = 0 \Longrightarrow$ $f^{2n}(0) = (-1)^n \quad , f^{(2n+1)} = 0$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$