Example:- Find the Maclaurian Series generated by $\sin x$ at $x = 0$

Solution:-

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^4 + \cdots$$

$f(x) = \sin x$ $\quad$ $f'(x) = \cos x$

$f''(x) = -\sin x$ $\quad$ $f'''(x) = -\cos x$

$f^4(x) = \sin x$ $\quad$ $f^5(x) = \cos x$

$f^{2n}(x) = (-1)^n \sin x$ $\quad$ $f^{(2n+1)}(x) = (-1)^n \cos x$

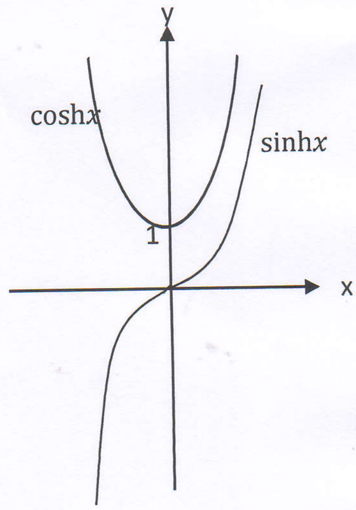
$f^{2n}(0) = 0$ $\quad$ $f^{(2n+1)}(0) = (-1)^n$

$$\sin x = 0 + \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example:- Find the Maclaurian Series generated by $\cosh x$ at $x = 0$.

Solution:-

Since $\quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2}$



$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots + \frac{x^n}{n!} + \cdots, \text{(from example pp} - 3)$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \ldots + \frac{x^n}{n!} + \cdots$$

so

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

H.W: Find the Maclaurian Series generated by $\sinh x$ at $x = 0$.

The Identity $e^{i\theta} = \cos\theta + i\sin\theta$

Since the Maclaurian Series generated by e^x is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots + \frac{x^n}{n!} + \cdots$$

now, replace x by $i\theta$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\,\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\,\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

$$e^{i\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}\cdots\right)}_{\sin\theta}$$

Taylor Series :-

If, instead of approximating the value of f near zero, we are concerned with values of x near some other point a, we write our approximating polynomials in powers of $(x-a)$:

$p_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

When we determine the coefficients $a_0, a_1, \ldots, a_n$, we are led to a series as follows :-

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \text{-----------}(8)$$

Which is called the **Taylor series** generated by f at $x = a$

Note :-

(1) Maclaurin Series is a Taylor Series with $a = 0$

(2) A function cannot generate a Taylor series expansion about $x = a$ unless it has finite derivatives of all order at $x = a$

Example :- Find the Taylor series generated by $\cos x$ at the point $a = 2\pi$

Solution:- The value of $\cos x$ and its derivative at $a = 2\pi$ are the same as their values at $a = 0$. Therefore

$f^{(2k)}(x) = (-1)^k \cos x,$ $\quad f^{(2k+1)}(x) = (-1)^{k+1} \sin x$

$f^{(2k)}(2\pi) = (-1)^k$ and $f^{(2k+1)}(2\pi) = 0$

as in previous example. The required series is

$$= 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{(x-2\pi)^{2k}}{(2k)!}$$

Example :- What are the first five terms of the Taylor series generated by $f(x) = \frac{1}{x}$ at $x = 2$?

Solution :- The first 5 terms are with $a = 2$

$$f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2!} + \frac{f^{(3)}(2)(x-2)^3}{3!} + \frac{f^{(4)}(2)(x-2)^4}{4!}$$

We have

$f(x) = x^{-1},$ $\quad f(2) = 2^{-1} = \frac{1}{2},$

$f'(x) = -x^{-2}$, $f'(2) = -\frac{1}{2^2}$,

$f''(x) = 2x^{-3}$, $f''(2) = \frac{2!}{2^3}$,

$f'''(x) = -6x^{-4}$, $f'''(2) = \frac{3!}{2^4}$,

$f^4(x) = +24x^{-5}$, $f^4(2) = -\frac{4!}{2^5}$,

The first, 5 terms of the Taylor series are,

$$\frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \frac{(x-2)^3}{2^4} + \frac{(x-2)^4}{2^5} + \cdots$$

The Binomial Series :-

The Maclaurin series generated by $f(x) = (1+x)^m$ is called Binomial Series:-

To derive the series, we first list the function and its derivative:-

$f(x) = (1+x)^m$

$f'(x) = m(1+x)^{m-1}$

$f''(x) = m(m-1)(1+x)^{m-2}$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$

⋮

$f^k(x) = m(m-1)(m-2)\ldots(m-k+1)(1+x)^{m-k}$

We then evaluate these at $x = 0$ and sub. In the Maclaurin series formula to obtain

$$= 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)(m-2)\ldots(m-k+1)x^k}{k!} \quad \text{------------------------(*)}$$

Example:-

Use the binomial series to estimate $\sqrt{1.25}$ with an error of less than 0.001.

Solution :-

We take $x = \frac{1}{4}, m = \frac{1}{2}$ in eq (*) to obtain

$$(1+\tfrac{1}{4})^{1/2} = 1 + \frac{1}{2}\frac{1}{4} + \frac{(1/2)(-1/2)}{2!}(\tfrac{1}{4})^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}(\tfrac{1}{4})^3 + \cdots$$

$$= 1 + \frac{1}{8} - \frac{1}{128} + \frac{1}{1024} - \frac{5}{32768} + \cdots$$

$\frac{1}{1024} = 0.00097 < 0.001$

For error less than 0.001 , $\Longrightarrow$ the approximation

$\sqrt{1.25} \cong 1 + \frac{1}{8} - \frac{1}{128} = 1.1171875$ is with $\frac{1}{1024}$ of the exact value and thus the required accuracy.

Convergence of Power Series :-

The power series

$$\sum_{n=0}^{\infty} a_n x^n$$

Defines a function whenever it converges, namely, the function f whose value at each x is the number.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Theorem :- The Converges Theorem for Power Series

If a power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 + a_2 x^2 + \cdots$$

Converges for $x = c\ (c \neq 0)$, then converges absolute for all $|x| < |c|$. If the series diverges for $x = d$ then it diverges for all $|x| > |d|$.

Finding the Interval of Convergence :-

The interval of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ can often be found by applying the **Ratio test** or the **Root test** to the series of absolute values, $\sum_{n=0}^{\infty} |a_n x^n|$

Thus, if

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}\, x^{n+1}}{a_n\, x^n} \right| \quad (Ratio\ Test) \qquad \rho = \lim_{n\to\infty} \sqrt[n]{|a_n x^n|} \quad (Root\ Test)$$

Then ,

(a) $\sum a_n x^n$ converges absolutely for all values of x for which $\rho < 1$

(b) $\sum a_n x^n$ diverges at all values of x for which $\rho > 1$,

(c) $\sum a_n x^n$ may either converges or diverges at a value of x for which $\rho = 1$

<u>Example :-</u> Find the interval of convergence of $\sum_{n=1}^{\infty}\frac{x^n}{n}$,

Solution :-We apply the Ratio Test to the series with absolute values and find

$$\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = |x|$$

Therefore the original series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$

<u>Example :-</u> For what values of x does the series $\sum_{n=1}^{\infty}\frac{(2x-5)^n}{n^2}$ converges ?

<u>Solution :-</u> we apply the Root Test to the series of absolute value

$$\rho = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(2x-5)^n}{n^2}\right|} = \lim_{n\to\infty}\frac{|2x-5|}{\sqrt[n]{n^2}} = \frac{|2x-5|}{1} = |2x-5|,$$

The series converges absolutely for

$|2x-5| < 1$ or $-1 < 2x-5 < 1$

or

$4 < 2x < 6$ or $2 < x < 3$

At the end points

When $= 2$, the series is $\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\right]$, Which converges

When $= 3$, the series is $\sum_{n=1}^{\infty}\left[\frac{(1)^n}{n^2}\right]$, which converges,

Therefore , the interval of convergence is $2 \le x \le 3$