

$$\int_d^{d+2p} f(t) dt = \frac{a_0}{2} \int_d^{d+2p} dt + a_1 \int_d^{d+2p} \cos \frac{\pi t}{p} dt + \dots + a_n \int_d^{d+2p} \cos \frac{n\pi t}{p} dt + \dots + b_1 \int_d^{d+2p} \sin \frac{\pi t}{p} dt + \dots + b_n \int_d^{d+2p} \sin \frac{n\pi t}{p} dt + \dots$$

The first term on the right hand side is simply $= \frac{1}{2} a_0 t \Big|_d^{d+2p} = p a_0$, and by equations (2) and (3) all integrals contains cosine and sine vanishes, then:-

$$a_0 = \frac{1}{p} \int_d^{d+2p} f(t) dt$$

To find a_n ($n = 1, 2, 3, \dots$) multiply both sides of equation (1) by $\cos \frac{n\pi t}{p}$ and then integrate from $t = d$ to $t = d + 2p$

$$\begin{aligned} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt &= \frac{a_0}{2} \int_d^{d+2p} \cos \frac{n\pi t}{p} dt + a_1 \int_d^{d+2p} \cos \frac{\pi t}{p} \cos \frac{n\pi t}{p} dt + \dots + a_n \int_d^{d+2p} \cos^2 \frac{n\pi t}{p} dt + \dots \\ &+ b_1 \int_d^{d+2p} \cos \frac{n\pi t}{p} \sin \frac{\pi t}{p} dt + \dots + b_n \int_d^{d+2p} \cos \frac{n\pi t}{p} \sin \frac{n\pi t}{p} dt + \dots \end{aligned}$$

By equations (2), (4) and (6) all terms on the right hand vanishes except the one involving $\cos^2 \frac{n\pi t}{p}$

$$\int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt = p a_n$$

$$a_n = \frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt$$

To find b_n ($n = 1, 2, 3, \dots$) multiply both sides of equation (1) by $\sin \frac{n\pi t}{p}$ and then integrate from $t = d$ to $t = d + 2p$

$$\begin{aligned}
 & \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt \\
 &= \frac{a_0}{2} \int_d^{d+2p} \sin \frac{n\pi t}{p} dt + a_1 \int_d^{d+2p} \cos \frac{\pi t}{p} \sin \frac{n\pi t}{p} dt + \dots \\
 &+ a_n \int_d^{d+2p} \sin \frac{n\pi t}{p} \cos \frac{n\pi t}{p} dt + \dots + b_1 \int_d^{d+2p} \sin \frac{\pi t}{p} \sin \frac{n\pi t}{p} dt + \dots \\
 &+ b_n \int_d^{d+2p} \sin^2 \frac{n\pi t}{p} dt + \dots
 \end{aligned}$$

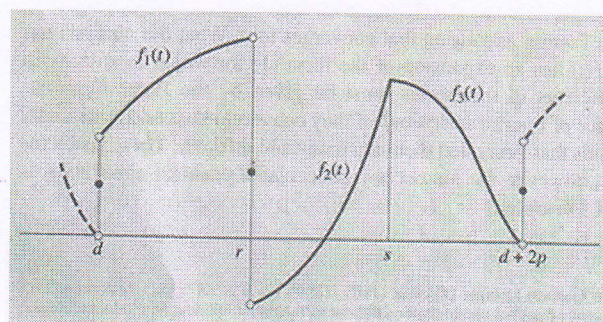
Similarly

$$b_n = \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt$$

a_0, a_n, b_n are called **Euler-Fourier** formulas, and the series (1) when its coefficients have these values is known the **Fourier Series of $f(t)$**

Dirichlet Theorem If $f(t)$ is a bounded periodic function which in any one periodic has at most a finite number of local maximum and minimum and a finite number of points of discontinuity, then the Fourier series of $f(t)$ converges to $f(t)$ at all points when $f(t)$ is continuous and converges to the average of the right- and left- hand limits of $f(t)$ at each point where $f(t)$ is discontinuous.

In Figure below the function $f(t)$ is defined by three different expressions $f_1(t)$, $f_2(t)$, and $f_3(t)$ successive portions of the period interval $d \leq t \leq t + 2p$. Hence the Euler formulas can be written as



$$a_n = \frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt = \frac{1}{p} \int_d^r f_1(t) \cos \frac{n\pi t}{p} dt + \frac{1}{p} \int_r^s f_2(t) \cos \frac{n\pi t}{p} dt + \frac{1}{p} \int_s^{d+2p} f_3(t) \cos \frac{n\pi t}{p} dt$$

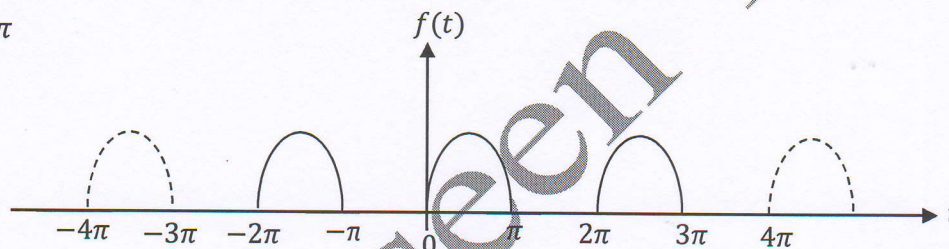
$$b_n = \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt = \frac{1}{p} \int_d^r f_1(t) \sin \frac{n\pi t}{p} dt + \frac{1}{p} \int_r^s f_2(t) \sin \frac{n\pi t}{p} dt + \frac{1}{p} \int_s^{d+2p} f_3(t) \sin \frac{n\pi t}{p} dt$$

Example:- What is the Fourier expansion of the periodic function whose definition in one period is

$$f(t) = \begin{cases} 0 & -\pi \leq t \leq 0 \\ \sin t & 0 \leq t \leq \pi \end{cases}$$

Solution

Graph the given function, then from the graph the half-period of the given function is $p = \pi$, taking $d = -\pi$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt = \frac{1}{\pi} \int_{-\pi}^0 0 \cdot \cos nt dt + \frac{1}{\pi} \int_0^{\pi} \sin t \cos nt dt$$

From identities

$$\sin t \cos nt = \frac{1}{2} [\sin(1+n)t + \sin(1-n)t]$$

$$a_n = \frac{1}{\pi} \left[-\frac{1}{2} \left\{ \frac{\cos(1-n)t}{1-n} + \frac{\cos(1+n)t}{1+n} \right\} \right]_0^{\pi} =$$

$$a_n = -\frac{1}{2\pi} \left[\frac{\cos(\pi-n\pi)}{1-n} + \frac{\cos(\pi+n\pi)}{1+n} - \left(\frac{1}{1-n} + \frac{1}{1+n} \right) \right] = -\frac{1}{2\pi} \left[\frac{-\cos n\pi}{1-n} + \frac{-\cos n\pi}{1+n} - \frac{2}{1-n^2} \right]$$

$$a_n = \frac{1+\cos n\pi}{\pi(1-n^2)} \quad n \neq 1$$

For $n = 1$ $a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos t dt = \frac{1}{\pi} \int_0^{\pi} \sin t \cos t dt = \frac{\sin^2 t}{2\pi} \Big|_0^{\pi} = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt = \frac{1}{\pi} \int_{-\pi}^0 0 \cdot \sin nt dt + \frac{1}{\pi} \int_0^{\pi} \sin t \sin nt dt$$

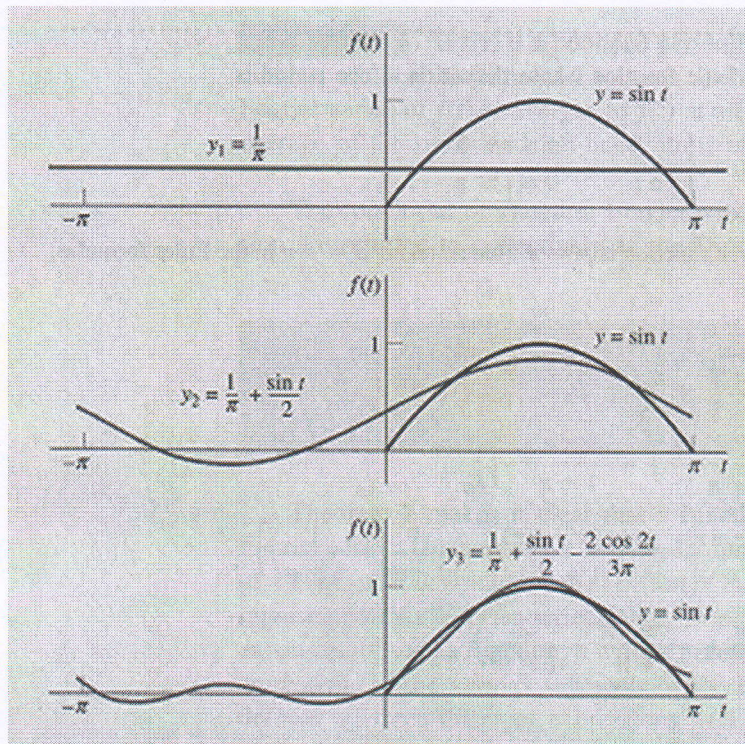
Since $\sin t \sin nt = \frac{1}{2} [-\cos(1+n)t + \cos(1-n)t]$

$$b_n = \frac{1}{\pi} \left[\frac{1}{2} \left\{ \frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1+n} \right\} \right]_0^{\pi} = 0 \quad n \neq 1$$

$$\text{For } n = 1 \quad b_1 = \frac{1}{\pi} \int_0^{\pi} \sin^2 t \, dt = \frac{1}{\pi} \int_0^{\pi} \frac{1 - \cos 2t}{2} \, dt = \frac{1}{\pi} \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{\pi} = \frac{1}{2}$$

Hence, evaluating the coefficients for $n = 0, 1, 2, 3, \dots$, we have

$$f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi} \left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \dots \right)$$



Theorem 1 If $f(t)$ is an even periodic function which satisfies the Dirichlet conditions, the coefficients in the Fourier series of $f(t)$ are given by the formulas

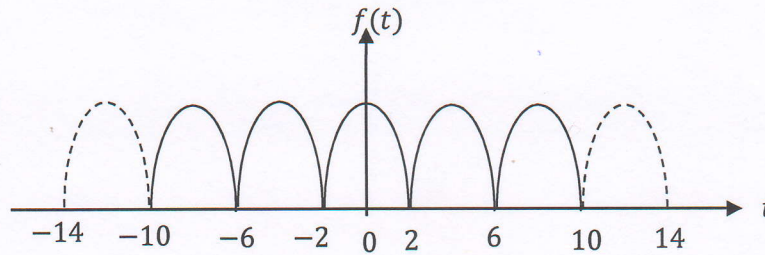
$$a_n = \frac{2}{p} \int_0^p f(t) \cos \frac{n\pi t}{p} \, dt \quad b_n = 0 \quad \text{where } 2p \text{ is the period of } f(t)$$

Example:- What is the Fourier expansion of the periodic function whose definition in one period is

$$f(t) = 4 - t^2 \quad -2 \leq t \leq 2$$

Solution

Graph the given function, then from the graph the half-period of the given function is $p = 2$, taking



Since $f(t) = f(-t)$ the given function is even function then $b_n = 0$

$$a_n = \frac{2}{2} \int_0^2 f(t) \cos \frac{n\pi t}{2} dt = \int_0^2 (4 - t^2) \cos \frac{n\pi t}{2} dt$$

$$a_n = \int_0^2 4 \cos \frac{n\pi t}{2} dt - \int_0^2 t^2 \cos \frac{n\pi t}{2} dt$$

$$a_n = \left[\frac{8}{n\pi} \sin \frac{n\pi t}{2} \right]_0^2 - \left[\frac{8t}{n^2\pi^2} \cos \frac{n\pi t}{2} + \left(\frac{2t^2}{n\pi} - \frac{16}{n^3\pi^3} \right) \sin \frac{n\pi t}{2} \right]_0^2$$

$$a_n = -\frac{16}{n^2\pi^2} \cos n\pi = \frac{16}{n^2\pi^2} (-1)^{n+1} n \neq 0 \quad \text{D} \quad \text{I}$$

For $n = 0$ $a_0 = \frac{2}{2} \int_0^2 (4 - t^2) dt = \left[4t - \frac{t^3}{3} \right]_0^2 = \frac{16}{3}$

t^2	+	$\cos \frac{n\pi t}{2}$
$2t$	-	$\frac{2}{n\pi} \sin \frac{n\pi t}{2}$
2	+	$-\left(\frac{2}{n\pi}\right)^2 \cos \frac{n\pi t}{2}$
0	+	$-\left(\frac{2}{n\pi}\right)^3 \sin \frac{n\pi t}{2}$

Substituting these coefficients into the series, we obtain

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2} \left(\cos \frac{\pi t}{2} - \frac{1}{2^2} \cos \frac{\pi t}{2} + \frac{1}{3^2} \cos \frac{\pi t}{2} - \frac{1}{4^2} \cos \frac{\pi t}{2} + \dots \right)$$

Theorem 2 If $f(t)$ is an odd periodic function which satisfies the Dirichlet conditions, the coefficients in the Fourier series of $f(t)$ are given by the formulas

$$b_n = \frac{2}{p} \int_0^p f(t) \sin \frac{n\pi t}{p} dt \quad a_n = 0 \quad \text{where } 2p \text{ is the period of } f(t)$$

Example:- What is the Fourier expansion of the periodic function whose definition in one period is

$$f(t) = t \quad -1 < t < 1$$

Solution