$$2 \le |z| \le 3$$

Solution

The annulus bounded by the circles with center at the origin and radii 2 and 3, including the boundary circles; bounded, closed, multiply connected.

Analytic Functions

The derivative of a function of a complex variable $w = f(z)$ is defined to be

$$\frac{dw}{dz} = w' = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad \cdots\cdots\cdots\cdots (1)$$

Familiar formulas of a complex variable derivative

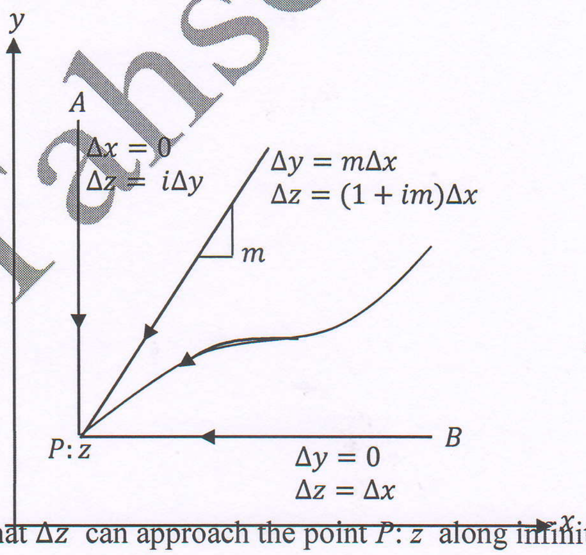
1- $\frac{d(w_1 \pm w_2)}{dz} = \frac{dw_1}{dz} \pm \frac{dw_2}{dz}$

2- $\frac{d(w_1 w_2)}{dz} = w_1 \frac{dw_2}{dz} + w_2 \frac{dw_1}{dz}$

3- $\frac{d(w_1/w_2)}{dz} = \frac{w_2(dw_1/dz) - w_1(dw_2/dz)}{w_2^2}$

4- $\frac{dw^n}{dz} = n w^{n-1} \frac{dw}{dz}$

Since $\Delta z = \Delta x + i\Delta y$ is itself a complex variable, the question is how it is approach zero?



From the Figure, it is clear that Δz can approach the point $P: z$ along infinity many different paths. In particular, Q can approach P along the line AP on which Δx is zero or along the line BP on which Δy is zero. Clearly, ***for the derivative of $f(z)$ to exist, it is necessary that the limit of the difference quotient (1) be the same no matter how Δz approaches zero.***

Cauchy-Riemann Equations

<u>Theorem</u>:- If u and v are real single - valued functions of x and y which, with their four first partial derivatives, are continuous throughout a region R, then the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are both necessary and sufficient conditions that $f(z) = u(x,y) + iv(x,y)$ be analytic in R. In this case, the derivative of $f(z)$ is given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{or} \qquad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

<u>Example</u>:-

For $(z) = \bar{z} = x - iy$, does $f'(z)$ exist ?

<u>Solution</u>

We have $u = x$ and $v = -y$,then

$$\frac{\partial u}{\partial x} = 1\,, \qquad \frac{\partial v}{\partial x} = 0 \qquad , \qquad \frac{\partial v}{\partial y} = -1 \,, \qquad \frac{\partial u}{\partial y} = 0$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ there is no point in the z-plane where $f'(z)$ exist.

<u>Example</u>:-

For $f(z) = z\bar{z}$, does $f'(z)$ exist ?

<u>Solution</u>

$f(z) = z\bar{z} = (x+iy)(x-iy) = x^2 + y^2$, then we have $u = x^2 + y^2$ and $v = 0$

$$\frac{\partial u}{\partial x} = 2x \qquad , \qquad \frac{\partial v}{\partial x} = 0 \qquad , \qquad \frac{\partial v}{\partial y} = 0\,, \qquad \frac{\partial u}{\partial y} = 2y$$

are continuous everywhere. However Cauchy-Riemann equations, which in this case are

$$2x = 0 \qquad \text{and} \qquad 2y = 0$$

are satisfied only at the origin. Hence $z = 0$ is the only point at which $f'(z)$ exist, and therefore $f(z) = z\bar{z}$ is nowhere analytic.

<u>Example</u>:-

For $f(z) = z^2$, does $f'(z)$ exist ?

<u>Solution</u>

$f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + 2ixy$, then we have $u = x^2 - y^2$ and $v = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad , \qquad \frac{\partial v}{\partial x} = 2y \qquad , \qquad \frac{\partial v}{\partial y} = 2x \qquad , \qquad \frac{\partial u}{\partial y} = -2y$$

are continuous everywhere, and Cauchy-Riemann equations, which in this case are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $f'(z)$ exist at all points of z-plane, and its value is

$$f'(z) = 2x + 2iy = 2z$$

The Elementary Functions of z

The exponential function e^z is of fundamental importance, not only for its own sake but also as a basis for defining all the other elementary transcendental functions.

Properties of e^z

1- $f(z) = w = e^z$

Let $z = x + iy$ $\quad f(z) = w = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) = u(x,y) + iv(x,y)$

Then $\quad u = e^x \cos y \qquad v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y\ , \qquad \frac{\partial v}{\partial y} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y \quad , \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

Since

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ then e^z is analytic everywhere

2- $f'(z) = \frac{de^z}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^z$

3- $\cos z = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

Example:- Prove that $\quad \cos^2 z + \sin^2 z = 1$

Solution

By substituting $\quad \cos z = \frac{e^{iz}+e^{-iz}}{2} \quad$ and $\quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

in the given formula

$$\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = 1$$

$$\frac{1}{4}\left(e^{2iz} + 2 + e^{-2iz}\right) - \frac{1}{4}\left(e^{2iz} - 2 + e^{-2iz}\right) = 1$$

$$1 = 1$$

Example:- Prove that $\quad \frac{d(\cos z)}{dz} = -\sin z$

Solution

Since $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ $\quad$ then $\quad \frac{d(\cos z)}{dz} = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) \cdot \frac{i}{i}$

$$\frac{d(\cos z)}{dz} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

Similarly we can prove

$$\frac{d(\sin z)}{dz} = \cos z$$

Example:- Prove that $\cos z = \cos x \cosh y - i \sin x \sinh y$

Solution

Let $z = x + iy$ $\qquad \cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}(e^{iz} + e^{-iz})$

$$\Rightarrow \quad \cos z = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) = \frac{1}{2}\, e^{-y}(\cos x + i \sin x) + \frac{1}{2}\, e^{y}(\cos x - i \sin x)$$

Or

$$\cos z = \cos x\, \frac{e^{y}+e^{-y}}{2} - i \sin x\, \frac{e^{y}-e^{-y}}{2}$$

$$\cos z = \cos(x + iy) = \cos x\ \cosh y - i \sin x\ \sinh y$$

Similarly we can prove

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

Now, in particular taking $x = 0$

$$\cos iy = \cosh y$$

$$\sin iy = i \sinh y$$

Example:- What is $\cos(1 + 2i)$

Solution

Let $x = 1$, $y = 2$ in the equation $\cos z = \cos(x + iy) = \cos x\ \cosh y - i \sin x\ \sinh y$

$$\cos(1 + 2i) = \cos 1\ \cosh 2 - i \sin 1\ \sinh 2 = (0.5403)(3.762) - i(0.8415)(3.627)$$

$$\cos(1 + 2i) = 2.033 - 3.052i$$

Example:- Prove that the only values for which only $\sin z = 0$ are the real number of $z = 0, \mp\pi, \mp 2\pi, \cdots$

Solution

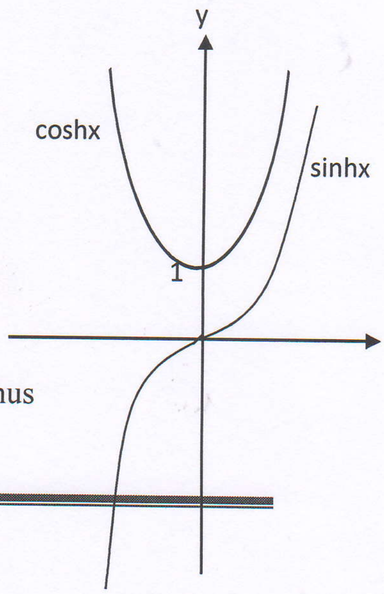
Since $\sin z = \sin x \cosh y + i \cos x \sinh y$

When $\sin z = 0 = \sin x \cosh y + i \cos x \sinh y$

$\sin x \cosh y = 0 \quad \cdots\cdots\cdots\cdots (1) \quad \cos x \sinh y = 0 \quad \cdots\cdots\cdots\cdots (2)$

Since $\cosh y \geq 1$ then equation (1) satisfies only if $\sin x = 0$, that is only

$$x = 0, \mp\pi, \mp 2\pi, \cdots$$



But for these values of x $\cos x$ either 1 or -1 and therefore cannot vanish. Thus for the second equation to hold, it is necessary that $\sinh y = 0 \Rightarrow y = 0$