hence the only values of z for which $\sin z = 0$ are of the form

$$z = n\pi + 0i = n\pi \quad \text{where} \quad n = 0, \mp 1, \mp 2, \cdots$$

Hyperbolic function

From the definition of hyperbolic function we know that

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

Let $z = x + iy$ then

$$\cosh z = \frac{e^{(x+iy)} + e^{-x-iy}}{2} = \frac{1}{2}(e^x \cos y + i\, e^x \sin y) + \frac{1}{2}(e^{-x} \cos y - i\, e^{-x} \sin y)$$

$$\cosh z = \frac{1}{2}(e^x + e^{-x}) \cos y + \frac{1}{2} i(e^x + e^{-x}) \sin y$$

Then $$\boxed{\cosh z = \cosh x \cos y + i \sinh x \sin y}$$

By the same way we can prove that

$$\boxed{\sinh z = \sinh x \cos y + i \cosh x \sin y}$$

In particular, setting $= 0$, we find

$$\cosh iy = \cos y$$

$$\sinh iy = i \sin y$$

The Logarithm of *z*

Let $w = \ln z$ then $z = e^w$ ………… (1)

If we let $w = u + iv$ and $z = re^{i\theta}$ then from equation (1)

$$e^{u+iv} = e^u e^{iv} = re^{i\theta}$$

Hence $e^u = r$, or $u = \ln r$ and $v = \theta$. Thus

$$w = u + iv = \ln r + i\,\theta$$

$$\ln z = \ln|z| + i \arg z$$

If we let θ_1 be the **principle argument** of z, the particular argument of z which lies in the interval $-\pi < \theta \leq \pi$. $\ln z$ can be written

$$\ln z = \ln|z| + i\,(\theta_1 + 2n\pi) \qquad n = 0, \pm 1, \pm 2, \pm 3\pi, \cdots\cdots$$

which shows that the logarithm function is infinitely many-valued. For any particular value of n a unique branch of the function is determined. If $n = 0$, the resulting branch of the logarithmic function is called **principle value.**

Theorem1

For every $n = 0, \pm 1, \pm 2, \pm 3, \cdots\cdots$ $\ln z$ is analytic except at 0 and on the negative real axis, and has derivative

$$\frac{d(\ln z)}{dz} = \frac{1}{z}$$

Theorem2

The principle value of $\ln z$ satisfies the following relations:

$$\ln z_1 z_2 = \begin{cases} \ln z_1 + \ln z_2 + 2i\pi & -2\pi < \arg z_1 + \arg z_2 \le -\pi \\ \ln z_1 + \ln z_2 & -\pi < \arg z_1 + \arg z_2 \le \pi \\ \ln z_1 + \ln z_2 - 2i\pi & \pi < \arg z_1 + \arg z_2 \le 2\pi \end{cases}$$

$$\ln \frac{z_1}{z_2} = \begin{cases} \ln z_1 - \ln z_2 + 2i\pi & -2\pi < \arg z_1 - \arg z_2 \le -\pi \\ \ln z_1 - \ln z_2 & -\pi < \arg z_1 - \arg z_2 \le \pi \\ \ln z_1 - \ln z_2 - 2i\pi & \pi < \arg z_1 - \arg z_2 \le 2\pi \end{cases}$$

$$\ln z^m = m \ln z - 2ki\pi$$

Where k is unique integer such that $(m/2\pi)\arg z - \frac{1}{2} \le k < (m/2\pi)\arg z + \frac{1}{2}$

* * *

General powers of z are defined by the formula

$$z^\alpha = e^{\alpha \ln z} = e^{\alpha\,[\ln|z| + i(\theta_1 + 2n\pi)]}$$

$$= e^{\alpha \ln|z|}\; e^{\alpha\theta_1 i}\; e^{2n\alpha\pi i}$$

Example:- What is the principle value of $(1+i)^i$

Solution

By definition

$$(1+i)^i = e^{\ln(1+i)^i} = e^{i\ln(1+i)}$$

$$= e^{i(\ln\sqrt{2} + i[(\pi/4)+2n\pi])} = e^{-[(\pi/4)+2n\pi]}\, e^{i\ln\sqrt{2}}$$

The principle value of this, obtained by taking $n = 0$,is

$$e^{-(\pi/4)}[\cos\,(\ln\sqrt{2}) + i\sin\,(\ln\sqrt{2})]$$

$$= e^{-0.7854}(\cos\,0.3466 + i\sin 0.3466\,) = \;0.429 + 0.155i$$

Example:- If $z_1 = i$ and $z_2 = -1 + i$, find the real and imaginary parts of $\ln z_1 z_2$

Solution

$\arg z_1 = \frac{\pi}{2}$ $\quad \arg z_2 = \frac{3\pi}{4}$ $\quad \arg z_1 + \arg z_2 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4} = 225^\circ$ $\quad 180^\circ < 225^\circ < 360^\circ$

Then $\ln z_1 z_2 = \ln z_1 + \ln z_2 - 2i\pi = \ln\,|z_1| + i\arg\,z_1 + \ln\,|z_2| + i\arg\,z_2 - 2i\pi$

$$\ln z_1 z_2 = \ln 1 + i\frac{\pi}{2} + \ln\sqrt{2} + i\frac{3\pi}{4} - 2i\pi = \ln\sqrt{2} + i\left(-\frac{3\pi}{4}\right)$$

The Inverse Trigonometric and Hyperbolic Functions

$$w = \cos^{-1} z \quad \cdots\cdots\cdots (1) \Rightarrow \quad z = \cos w$$

then $z = \frac{e^{iw}+e^{-iw}}{2}$ $\Rightarrow$ $2z = e^{iw} + e^{-iw}$

multiply both sides by e^{iw} $e^{2iw} - 2ze^{iw} + 1 = 0$

$\therefore$ $e^{iw} = z \pm \sqrt{z^2 - 1}$

Taking logarithm of both sides

$$iw = \ln\ [z \pm \sqrt{z^2 - 1}]\ \cdot \frac{i}{i}$$

$$w = -i\ \ln\ [z \pm \sqrt{z^2 - 1}]$$

From equation (1)

$$\boxed{\cos^{-1} z = -i\ \ln\ [z \pm \sqrt{z^2 - 1}]}$$

Example:- Prove that $\sin^{-1} z = -i \ln\ [iz \pm \sqrt{1 - z^2}]$

Solution

$$w = \sin^{-1} z \quad \cdots\cdots\cdots (1) \Rightarrow \quad z = \sin w$$

then $z = \frac{e^{iw}-e^{-iw}}{2i}$ $\Rightarrow$ $2zi = e^{iw} - e^{-iw}$

multiply both sides by e^{iw} $e^{2iw} - 2ize^{iw} - 1 = 0$

$\therefore$ $e^{iw} = iz \pm \sqrt{1 - z^2}$

Taking logarithm of both sides

$$iw = \ln\ [iz \pm \sqrt{1 - z^2}]$$

then $w = -i \ln\ [iz \pm \sqrt{1 - z^2}]$

From equation (1)

$$\boxed{\sin^{-1} z = -i \ln\ [iz \pm \sqrt{1 - z^2}]}$$

Example:- Prove that $\tan^{-1} z = \frac{i}{2} \ln \frac{i+z}{i-z}$

Solution let $w = \tan^{-1} z$ $\cdots\cdots\cdots (1) \Rightarrow$ $z = \tan w = \frac{\sin w}{\cos w}$

or $z = \frac{\frac{e^{iw}-e^{-iw}}{2i}}{\frac{e^{iw}+e^{-iw}}{2}}$ $\Rightarrow$ $iz = \frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} \cdot \frac{e^{iw}}{e^{iw}}$

$$iz = \frac{e^{i2w}-1}{e^{i2w}+1} \quad \Rightarrow \quad ize^{i2w} + iz = e^{i2w} - 1$$

$$e^{i2w} = \frac{iz+1}{1-iz} \cdot \frac{i}{i} \quad \Rightarrow \quad e^{i2w} = \frac{i-z}{i+z}$$

or $$i2w = \ln\frac{i-z}{i+z} \quad \Rightarrow \quad w = \frac{1}{2i}\ln\frac{i-z}{i+z} \cdot \frac{i}{i}$$

$$w = \tan^{-1}z = \frac{i}{2}\ln\left(\frac{i-z}{i+z}\right)^{-1}$$

then $$\boxed{\tan^{-1}z = \frac{i}{2}\ln\frac{i+z}{i-z}}$$

H.W Prove that

1- $\cosh^{-1}z = \ln(z \pm \sqrt{z^2-1})$

2- $\sinh^{-1}z = \ln(z \pm \sqrt{z^2+1})$

3- $\tanh^{-1}z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$

Example:- Prove that $\overline{\cos z} = \cos\bar{z}$

Solution

let $z = x + iy$ and $\bar{z} = x - iy$

since $\cos z = \cos(x+iy) = \cos x\ \cosh y - i\sin x\ \sinh y$

then $\overline{\cos z} = \cos x\ \cosh y + i\sin x\ \sinh y$

but $\cos\bar{z} = \cos\ (x - iy) = \cos x\ \cosh(-y) - i\sin x\ \sinh(-y)$

from properties of $\cosh(-y) = \cosh(y)$ $\sinh(-y) = -\sinh(y)$

$$\cos\bar{z} = \cos\ (x-iy) = \cos x\ \cosh y + i\sin x\ \sinh y = \overline{\cos z}$$

Example:- Express $\tan i$ in the form of $a + ib$, giving only principle values.

Solution

Since $\tan z = \frac{\sin z}{\cos z}$

But $\cos z = \cos(x+iy) = \cos x\ \cosh y - i\sin x\ \sinh y$

$\sin z = \sin(x+iy) = \sin x \cosh y + i\cos x \sinh y$

Now let $z = 0 + i$

then $$\tan i = \frac{\sin 0 \cosh 1 + i\cos 0 \sinh 1}{\cos 0\ \cosh 1 - i\sin 0\ \sinh 1} = i\ \frac{\sinh 1}{\cosh 1} = 0.762i$$

Example:- Find all solutions of equation $\cosh z = -2$

Solution