Complex Variable

Consider the number of the form $z = x + iy$ where x is real part of z and y is imaginary part of z. then z is called complex number

i is imaginary unit , $i^2 = -1$

Notes:-

1- Two complex numbers $a + ib$ and $c + id$ are said to be equal if and only if the real and imaginary part of the first are equal to the real and imaginary parts of the second.

Example:- If $(x + y + 2) + (x^2 + y)i = 0$, then $x + y + 2 = 0$ and $x^2 + y = 0$

from these pair of simultaneous equations $x^2 - x - 2 = 0$ $\Rightarrow$ $(x - 2)(x + 1) = 0$

then $x = 2$, and $y = -4$ or $x = -1$, and $y = -1$

2- The **conjugate** of a complex number z is $\overline{z}$ where $\overline{z} = x - iy$

3- $i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = 1$

$i^5 = i^4 \cdot i = i$

4- $\frac{1}{i} = -i$

5- **Addition , Subtraction**

$$(a + ib) \pm (c + id) = (a \pm c) + (b \pm d)i$$

6- **Multiplication**

$$(a + ib)(c + id) = (ac - bd) + (bc + ad)i$$

7- **Division**

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i$$

8- $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$ where $|z| = \sqrt{(\text{Re}z)^2 + (\text{Im}z)^2}$

9- $z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\,\text{Re}(z)$ $\Rightarrow$ $\text{Re}(z) = \frac{z+\overline{z}}{2}$

10- $z - \overline{z} = (x + iy) - (x - iy) = 2iy = 2i\,\text{Im}(z)$ $\Rightarrow$ $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

11- $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$

12- $\overline{z_1 z_2} = \overline{z}_1\, \overline{z}_2$

13- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

PROBLMES

1- Verify that $z = (1 + i\sqrt{3})/2$ satisfies the equation $z^2 - z + 1 = 0$

2- What is $\text{Re}(z^3 - 2z)$? What is $\text{Im}\,(z^3 - 2z)$

3- Reduce the following expression to the form $a + ib$

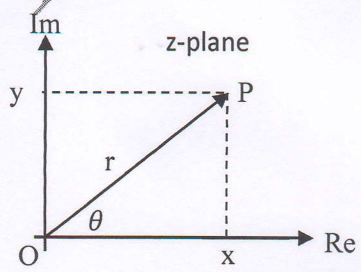
$$i(2 + 3i)^4$$

The Graphical Representation of Complex Number

Let $z = x + iy$ is a complex number then the vector OP represent the complex number z in z-plane

$r = \sqrt{x^2 + y^2} = |z| =$ **Absolute** or **modulus** value of z

$\theta = \tan^{-1}\frac{y}{x} =$ Argument of z (written as arg z)



and from the figure we see that

$$z = x + iy = r\cos\theta + i\,r\sin\theta$$

or $z = r\,(\cos\theta + i\ \sin\theta) \Rightarrow$ **polar** or **trigonometric** form of z

now, if we have two complex number

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1z_2 = [r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)] =$$
$$r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i\,(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

From the trigonometric identities $\boxed{z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + \text{i}\sin(\theta_1 + \theta_2)]}$

Similarly $z_1z_2z_3 = r_1r_2r_3[\cos(\theta_1 + \theta_2 + \theta_3) + \text{i}\sin(\theta_1 + \theta_2 + \theta_3)]$

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$$z_1z_2z_3\cdots z_n = r_1r_2r_3\cdots r_n[\cos(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n) + \text{i}\sin(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n)]$$

if all these z's are the same, then:

$$z^n = r^n\,(\cos n\theta + i\ \sin n\theta)$$

and if $|z| = 1$

$\boxed{z^n = \cos n\theta + i\ \sin n\theta}$ $\Rightarrow$ Demoiver's theorem if $\quad n = -1$

$$z^{-1} = \frac{1}{z} = \frac{1}{r}(\cos\theta - i\ \sin\theta)$$

The quotient of two complex number can be written

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{r_2(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 - i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta + \sin^2\theta}$$

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\ (\theta_1 - \theta_2) + i\ \sin(\theta_1 - \theta_2)\right]}$$

Example:- Using Demoiver's theorem and Binomial expansion , express $\cos 4\theta$ and $\sin 4\theta$ interms of powers of $\cos\theta$ and $\sin\theta$

Solution

From Demoiver's theorem let $n = 4$

$$z^4 = (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\ \sin 4\theta$$

remember that

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots + \frac{k(k-1)(k-2)\cdots(k-m+1)x^k}{k!} + \cdots$$

then $\quad (\cos\theta + i\sin\theta)^4 = \frac{\cos^4\theta}{\cos^4\theta}(\cos\theta + i\sin\theta)^4 = \cos^4\theta\left(1 + i\frac{\sin\theta}{\cos\theta}\right)^4$

now, let $k = 4$ and $x = i\frac{\sin\theta}{\cos\theta}$ in the Binomial series

$$\cos 4\theta + i\ \sin 4\theta =$$

$$\cos^4\theta\left[1 + 4\cdot i\frac{\sin\theta}{\cos\theta} + \frac{4(4-1)}{2!}\cdot\left(i\frac{\sin\theta}{\cos\theta}\right)^2 + \frac{4(4-1)(4-2)}{\cdot 3!}\left(i\frac{\sin\theta}{\cos\theta}\right)^3 + \frac{4(4-1)(4-2)(4-3)}{\cdot 4!}\left(i\frac{\sin\theta}{\cos\theta}\right)^4\right]$$

or $\quad \cos 4\theta + i\sin 4\theta = (\cos^4\theta - 6\cos^2\theta\ \sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\ \sin\theta - 4\sin^3\theta\ \cos\theta)$

Then $\quad \cos 4\theta = \cos^4\theta - 6\cos^2\theta\ \sin^2\theta + \sin^4\theta \quad$ and $\quad \sin 4\theta = 4\cos^3\theta\ \sin\theta - 4\sin^3\theta\ \cos\theta$

Roots

Equation $\quad z^n = r^n(\cos n\theta + i\ \sin n\theta)\quad$ can be extended to find the roots of integral orders

Let n - th roots of $z = r(\cos\theta + i\sin\theta)$ is defined by number $w = R(\cos\emptyset + i\sin\emptyset)$

then, $w^n = z$ or

$$R^n(\cos n\emptyset + i\sin n\emptyset) = r(\cos\theta + i\sin\theta)$$

comparing the two sides of this equation $R^n = r \quad \Rightarrow \quad R = r^{1/n}$

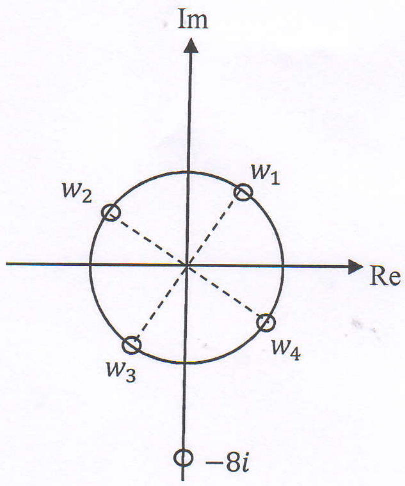
and the angles of equal complex numbers must either be equal or differ by an integral multiple of 2π

$$n\emptyset = \theta + 2k\pi \quad \text{or} \quad \emptyset = \frac{\theta + 2k\pi}{n} \quad \text{where} \quad k = 0, 1, 2, \cdots, n-1$$

$$\boxed{w = z^{1/n} = r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)}$$

Example:- Find the four fourth roots of $-8i$

Solution



$$z = 0 - 8i \qquad |z| = \sqrt{0^2 + (-8)^2} = 8$$

$$\theta = 270° = \frac{3\pi}{2} \qquad n = 4$$

Then $R = 8^{1/4}$

$$w = 8^{1/4}\left(\cos\frac{\frac{3\pi}{2} + 2k\pi}{4} + i\sin\frac{\frac{3\pi}{2} + 2k\pi}{4}\right)$$

$k = 0 \qquad w_1 = 8^{1/4}\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$

$k = 1 \qquad w_2 = 8^{1/4}\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$

$k = 2 \qquad w_3 = 8^{1/4}\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$

$k = 3 \qquad w_4 = 8^{1/4}\left(\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}\right)$

With integral powers and roots defined, the general rational power of complex number can be defined as

$$z^{p/q} = \left(z^{1/q}\right)^p = \left[r^{1/q}\left(\cos\frac{\theta + 2k\pi}{q} + i\sin\frac{\theta + 2k\pi}{q}\right)\right]^p$$

$$= r^{p/q}\left[\cos\frac{p}{q}(\theta + 2k\pi) + i\sin\frac{p}{q}(\theta + 2k\pi)\right] \qquad k = 0, 1, 2, \cdots, n-1$$

Example:- Find all the distinct values of $(-1 - i)^{4/5}$

Solution