$z = -1 - i \qquad |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\frac{-1}{-1} = 225° = \frac{5\pi}{4} \qquad q = 5 \qquad p = 4$

$z^{4/5} = (-1-i)^{4/5} = r^{4/5}\left[\cos\frac{4}{5}(\theta + 2k\pi) + i\sin\frac{4}{5}(\theta + 2k\pi)\right]$

$k = 0 \qquad = 2^{2/5}[\cos\pi + i\sin\pi]$

$k = 1 \qquad = 2^{2/5}\left[\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right]$

$k = 2 \qquad = 2^{2/5}\left[\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right]$

$k = 3 \qquad = 2^{2/5}\left[\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right]$

$k = 4 \qquad = 2^{2/5}\left[\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right]$

Problems

1- Find all distinct cube roots of $1 + i$ and reduce each to the form $a + ib$

2- Find all the distinct values of $(1 - i)^{5/4}$

3- Using Demoiver's theorem and Binomial expansion, express $\cos 5\theta$ and $\sin 5\theta$ interms of powers of $\cos\theta$ and $\sin\theta$.

Absolute Values

The absolute of a complex number z is already defined to be the length of the vector which represents z, or

$$|z| = \sqrt{x^2 + y^2} = \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2}$$

since both $[\text{Re}(z)]^2$ and $[\text{Im}(z)]^2$ are nonnegative real numbers, then

1- $|z| \geq \text{Re}(z)$

2- $|z| \geq \text{Im}(z)$

3- $|z| = |\bar{z}|$

4- $|z\bar{z}| = |z|^2$

5- $|z_1 z_2| = |z_1||z_2|$

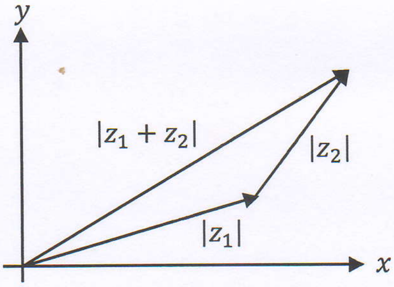
6- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

7- From the geometric addition of complex numbers Figure (a)

$|z_1 + z_2| \leq |z_1| + |z_2|$

For three terms $\qquad |z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$

For n – terms $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$

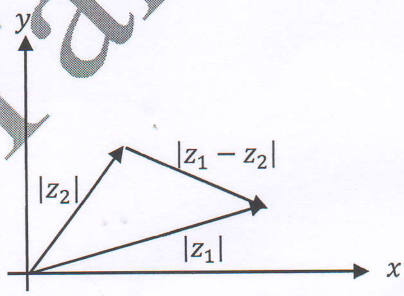


(a)

8- From the geometric subtraction of complex numbers Figure (b)

$$|z_1 - z_2| \geq \big||z_1| - |z_2|\big| \geq 0$$

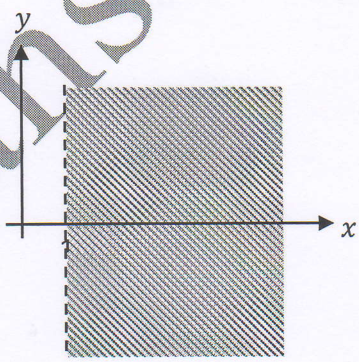
the outer absolute – value signs on right hand side for $|z_1| \geq |z_2|$



(b)

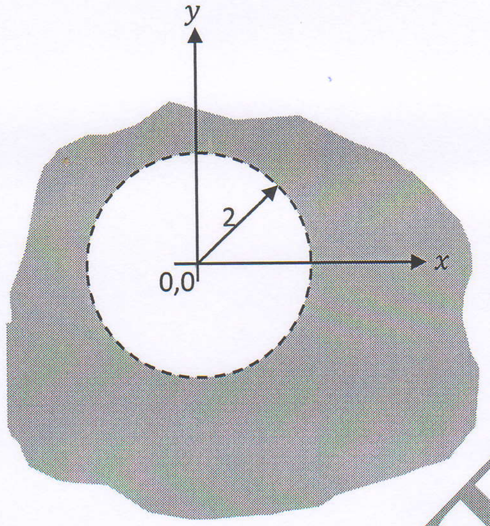
Example:- Describe the region in the z-plane defined by the inequality $\text{Re}(z) > 1$

Solution The given inequality defines the set of all points in the half plane to the right of the line $x = 1$



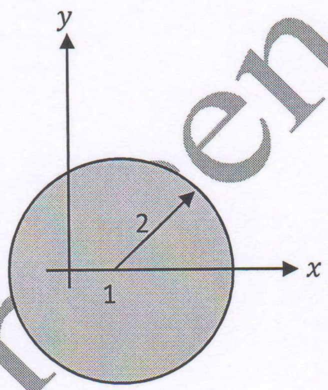
Example:- Describe the region in the z-plane defined by the inequality $|z| > 2$.

Solution Let $z = x + iy$ then $|z| = \sqrt{x^2 + y^2} > 2$ or $x^2 + y^2 > 2^2$ then the given inequality defines the set of all points lies outside the circle of radius 2 with center at origin



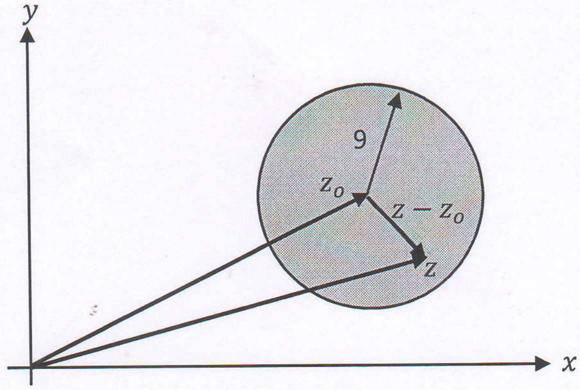
Example:- Describe the region in the z-plane defined by the inequality $|z-1| \leq 2$.

Solution



Example:- What region in the z-plane defined by the inequality $|z-z_o| \leq 9$

Solution The given inequality defines the set of all points within and on the circumference of the circle of the radius 9 which has the image of z_o as its center.



PROBLEMS

1- What region in the z-plane is defined by the inequalities $0 < \mathrm{Re}(z) \le \mathrm{Im}(z)$
2- What region in the z-plane is defined by the inequalities $|z-1| \le \mathrm{Re}(z)$
3- What region in the z-plane is defined by the inequalities $|z-1| + |z+1| > 3$
4- If $w = (3z+i)]/(i-z)$, show that, $\mathrm{Re}(z) \ge 0$ implies $\mathrm{Im}(w) \le 0$
5- If $w = (z+3)]/(z+2)$, show that, $\mathrm{Im}(z) \ge 0$ implies $\mathrm{Im}(w) \le 0$
6- If $w = [i(1-z)]/(1+z)$, prove that, $|z| < 1$ implies $\mathrm{Im}(w) > 0$

Function of Complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of z in some portion of the complex plane, one or more values of w are defined, then w is said to be a function of z and written as

$$w = f(z)$$

The assertion that w is a function of $z = x + iy$ can also be written

$$w = u(x, y) + iv(x, y)$$

Example:- If $w = f(z) = (x^2 - y) + i(x + y^2)$

Now, if $= 1 + 2i$, then $x = 1$ and $y = 2$, then

$$f(1+2i) = (1^2 - 2) + i(1 + 2^2) = -1 + 5i$$

Example:- If $w = f(z) = \frac{1}{z}$, write $f(z)$ in the form $u(x, y) + iv(x, y)$

Solution

By multiplying $f(z)$ by $\frac{\bar{z}}{\bar{z}}$ where $z = x + iy$ and $\bar{z} = x - iy$

$$f(z) = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} \qquad f(z) = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

then, $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = -\frac{y}{x^2+y^2}$

* * *

On the other hand, it may be impossible to express w in a form involving only the explicit combination $x + iy$ without using such "artificial" expressions as $\mathrm{Re}(z) \equiv x$ and $\mathrm{Im}(z) \equiv y$

Example:-

$$w = 7x + 3iy = 4\,\mathrm{Re}(z) + 3z = 7z - 4i\,\mathrm{Im}(z) = 5z + 2\bar{z}$$

Example:- Describe of the following set of points telling whether it is bounded or unbounded, open or closed, and simply or multiply connected.