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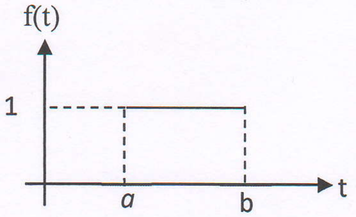
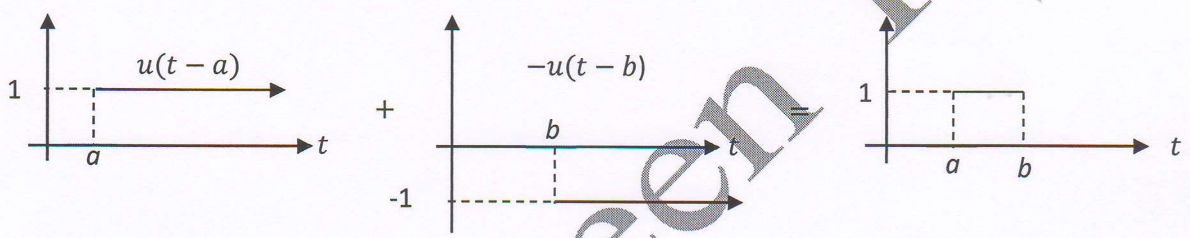
Multiplying the Function by Unit Step Function

Example:- What is the equation of the function whose graph is



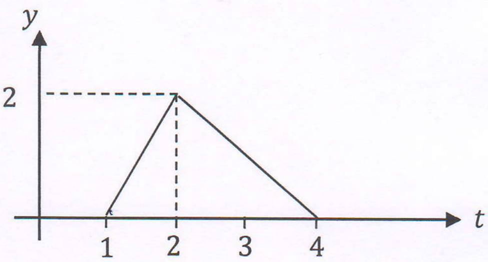
Solution

This function can be regarded as the sum of two translated (shifted) unit step functions as

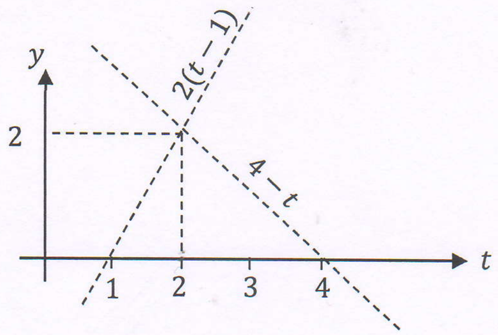


More generally, the expression $f(t-a)\cdot u(t-a)$ represents the function obtained by translating $f(t)$ a units to the right and cutting it off, i.e, making vanish identically to the left.

Example: What is the equation of the function whose graph is



Solution



the general equation of line is $y = mt + c$ where m is the slope of line then, for the left hand line $m = 2$ $y(t) = 2t + c$ $\cdots\cdots\cdots\cdots (a)$

and for right line $m = -1$ $y(t) = -t + c$ $\cdots\cdots\cdots\cdots (b)$

we need only one point to find c,

for left hand line when $t = 1$, $y = 0$ from Eq. (a) $c = -2$ so $y(t) = 2(t-1)$ for $1 \le t \le 2$

for right hand line when $t = 4$, $y = 0$ from Eq. (b) $c = 4$ so $y(t) = 4 - t$ for $2 \le t \le 4$

now, from $1 \le t \le 2$

$2(t-1)[u(t-1) - u(t-2)]$ {defines the segment of the given function between $t = 1$ and $t = 2$ and vanishes elsewhere}

now, from $2 \le t \le 4$

$(4-t)[u(t-2) - u(t-4)]$ {defines the segment of the given function between $t = 2$ and $t = 4$ and vanishes elsewhere}

so the function from $t = 1$ to $t = 4$ is

$$2(t-1)[u(t-1) - u(t-2)] + (4-t)[u(t-2) - u(t-4)]$$

or

$$2(t-1)u(t-1) - 2(t-1)u(t-2) + (4-t)u(t-2) - (4-t)u(t-4)$$

but

$$-2(t-1)u(t-2) + (4-t)u(t-2) = -2tu(t-2) + 2u(t-2) + 4u(t-2) - tu(t-2)$$

$$= 6\,u(t-2) - 3\,tu(t-2) = -3(t-2)u(t-2)$$

then, the function of graph is

$$2(t-1)u(t-1) - 3(t-2)u(t-2) + (t-4)u(t-4)$$

Second Shifting Theorem (t-Shifting)

L.T of $\{f(t-a)u(t-a)\} = e^{-as}$ L.T of $\{f(t)\}$

Prove By definition we have

L.T of $\{f(t-a)u(t-a)\} = \int_0^\infty \{f(t-a)u(t-a)\}\ e^{-st}\ dt$

because $u(t-a)$ vanishes the $f(t-a)$ identically to the left of $t = a$ then the integration will starts from a

or $\int_0^\infty \{f(t-a)u(t-a)\}\ e^{-st}\ dt = \int_a^\infty \{f(t-a)\}\ e^{-st}\ dt$

now by transformation of the domain of integration by letting $t = T + a \Rightarrow dt = dT$

note that $t = a$, $T = 0$ and when $t \to \infty$, $T \to \infty$

$$\int_a^{\infty}\{f(t-a)\}\, e^{-st}\; dt = \int_0^{\infty}\{f(T)\}\, e^{-s(T+a)}\; dT$$
$$= e^{-as}\int_0^{\infty}\{f(T)\}\, e^{-sT}\; dT = e^{-as}\ \text{L.T of}\ \{f(t)\}$$

Corollary 1:-

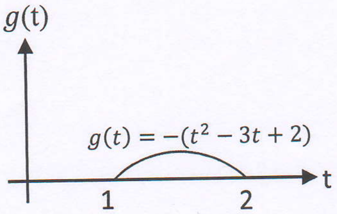
$$\text{L.T of}\{f(t)u(t-a)\} = e^{-as}\ \text{L.T of}\ \{f(t+a)\}$$

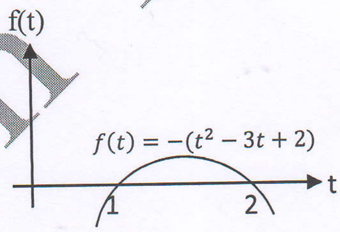
Corollary 2:-

if LT^{-1} of $\{\phi(s)\} = f(t)$, then LT^{-1} of $\{e^{-as}\phi(s)\} = f(t-a)u(t-a)$

This corollary states that suppressing the factor e^{-as} in transform requires that the inverse of what remains be translated a units to the right and cut off to the left of the point $t = a$

Example:- What is the transform of the function whose graph is shown in Figure





Solution

$$g(t) = f(t)u(t-1) - f(t)\,u(t-2)$$

where $f(t) = -(t^2 - 3t + 2)$

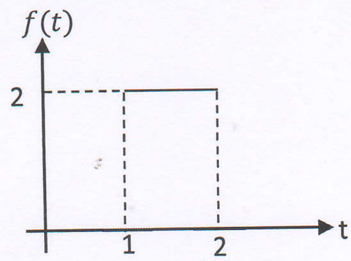
using Corollary 1, observing that $f(t+1) = -[(t+1)^2 - 3(t+1) + 2] = -(t^2 - t)$

and $f(t+2) = -[(t+2)^2 - 3(t+2) + 2] = -(t^2 + t)$

the required transform is

$$-e^{-s}\ \text{L.T of}\ \{t^2 - t\} + e^{-2s}\ \text{L.T of}\ \{t^2 + t\} = -e^{-s}\left(\frac{2}{s^3} - \frac{1}{s^2}\right) + e^{-2s}\left(\frac{2}{s^3} - \frac{1}{s^2}\right)$$

Example:- Find the solution of equations $y'(t) + 3y(t) + 2\int_0^t y\, dt = f(t)$ for which $y_0 = 1$ if $f(t)$ is the function whose graph is shown in Figure



Solution

In this case $f(t) = 2u(t-1) - 2u(t-2)$ then the differential equation can be written as

$$y'(t) + 3y(t) + 2\int_0^t y\,dt = 2u(t-1) - 2u(t-2)$$

now taking L.T of both sides we have

$$[s\,Y(s) - (1)] + 3Y(s) + 2\frac{1}{s}Y(s) = \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s}$$ Note :- since $a = 0$, then $\int_a^0 y(t)\,dt = 0$

$$(s^2 + 3s + 2)Y(s) = 2e^{-s} - 2e^{-2s} + s$$

or $$Y(s) = \frac{s}{(s+1)(s+2)} + \frac{2e^{-s}}{(s+1)(s+2)} - \frac{2e^{-2s}}{(s+1)(s+2)}$$

the first term in $Y(s)$ can be written as $\frac{s}{(s+1)(s+2)} = \frac{k_1}{s+1} + \frac{k_2}{s+2} \Rightarrow k_1 = -1$

and $k_2 = 2$

then $\frac{s}{(s+1)(s+2)} = \frac{2}{s+2} - \frac{1}{s+1}$ so $\text{LT}^{-1}\frac{s}{(s+1)(s+2)} = 2e^{-2t} - e^{-t}$

and by suppressing the exponential factor in the second term $Y(s)$

$$\frac{2}{(s+1)(s+2)} = \frac{k_1}{s+1} + \frac{k_2}{s+2} \Rightarrow k_1 = 2 \text{ and } k_2 = -2$$

then $\text{L.T}^{-1}\frac{2e^{-s}}{(s+1)(s+2)} = 2\left(e^{-(t-1)} - e^{-2(t-1)}\right)u(t-1)$

and $\text{L.T}^{-1}\frac{2e^{-2s}}{(s+1)(s+2)} = 2\left(e^{-(t-2)} - e^{-2(t-2)}\right)u(t-2)$

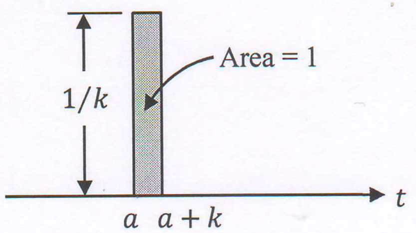
$$y(t) = 2e^{-2t} - e^{-t} + 2\left(e^{-(t-1)} - e^{-2(t-1)}\right)u(t-1) - 2\left(e^{-(t-2)} - e^{-2(t-2)}\right)u(t-2)$$

Dirac's Delta Function

Consider the function

$$f_k(t-a) = \begin{cases} 1/k & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases} \quad \cdots\cdots\cdots (1)$$

This function represents a force of magnitude $1/k$ acting from $t=a$ to $t=a+k$, where k is positive and small. The integral of a function acting over a time interval $a \le t \le a+k$ is called the **impulse** of the function.



Now, the impulse of f_k is

$$I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1 \quad \cdots\cdots\cdots (2)$$

By taking the limit of f_k as $k \to 0$

$$\lim_{k\to 0} f_k(t-a) = \delta(t-a)$$

$\delta(t-a)$ is called **Dirac delta function**

Note :- From equations (1) and (2) by taking limit as $k \to 0$ we obtain

$$\delta(t-a) = \begin{cases} \infty & if \ t=a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-a)\,dt = 1$$

Sifting property of $\delta(t-a)$

$$\boxed{\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)}$$

To obtain the L.T of $\delta(t-a)$, we write

$$f_k(t-a) = \frac{1}{k}[u(t-a) - u(t-(a+k))]$$

and take L.T $\quad$ L.T of $\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\frac{1-e^{-ks}}{ks}$

now, taking the limit as $k \to 0$ (using l'Hopital's rule)

$$\lim_{k\to 0} \frac{se^{-(a+k)s}}{s} = e^{-as}$$

Then $\quad \boxed{\text{L.T of } \delta(t-a) = e^{-as}}$