Example:- Evaluate $\int_0^\infty \sqrt{z} e^{-z^3} dz$

Solution let $t = z^3 \quad \Rightarrow \quad z = t^{1/3}$

$$dt = 3z^2 dz \quad \Rightarrow \quad dz = \frac{1}{3} z^{-2} dt$$

then $\int_0^\infty \sqrt{z} e^{-z^3} dz = \int_0^\infty \sqrt{t^{1/3}} e^{-t} \ \frac{1}{3} t^{-\frac{2}{3}} \ dt \ = \ \frac{1}{3}\int_0^\infty e^{-t} \ t^{-\frac{1}{2}-1+1} \ dt$

$$\therefore \quad \int_0^\infty \sqrt{z} e^{-z^3} dz = \frac{1}{3}\int_0^\infty e^{-t} \ t^{\frac{1}{2}-1} \ dt = \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

Euler Beta Function

The Beta function $\beta(x, y)$ is defined by the integral

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \qquad x > 0 \ , \qquad y > 0$$

The Euler Beta function can be represented in terms of Gamma function as:-

$$\boxed{\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$$

Example:- Evaluate $\int_0^1 t^{0.3} \ (1-t)^{0.9} dt$

Solution

$$\int_0^1 t^{1.3-1} \ (1-t)^{1.9-1} dt = \beta(1.3,1.9) = \frac{\Gamma(1.3)\Gamma(1.9)}{\Gamma(1.3+1.9)} = \frac{\Gamma(1.3)\Gamma(1.9)}{\Gamma(3.2)}$$

but $1.2(1.2+1)\Gamma(1.2) = \Gamma(1.2+1+1) = \Gamma(3.2)$

so $\Gamma(3.2) = 1.2 \cdot 2.2 \cdot 0.9182 = 2.424$

$$\therefore \quad \int_0^1 t^{0.3} \ (1-t)^{0.9} dt = \frac{0.8975 \cdot 0.962}{2.424} = \ 0.3562$$

Example:- Evaluate $\int_0^\infty \frac{y^{a-1}}{1+y} dy$

Solution $\int_0^\infty \frac{y^{a-1}}{1+y} \ dy = \int_0^\infty y^{a-1}(1+y)^{-1} \ dy$

Let $y = \frac{x}{1-x}$ where $y = 0$ when $x = 0$ and $y \to \infty$ when $x = 1$

$$\Rightarrow \quad dy = \frac{1-x+x}{(1-x)^2} dx = \frac{dx}{(1-x)^2}$$

Then the given integral becomes to

$$\int_0^{\infty}\frac{y^{a-1}}{1+y}dy = \int_0^1\left(\frac{x}{1-x}\right)^{a-1}\left(1+\frac{x}{1-x}\right)^{-1}(1-x)^{-2}dx$$

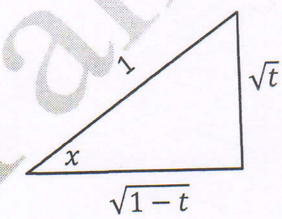
$$= \int_0^1\left(\frac{x}{1-x}\right)^{a-1}\left(\frac{1}{1-x}\right)^{-1}(1-x)^{-2}dx = \int_0^1 x^{a-1}(1-x)^{-(a-1)}(1-x)^{-1}dx$$

$$= \int_0^1 x^{a-1}(1-x)^{-a}dx = \int_0^1 x^{a-1}(1-x)^{-a+1-1}dx$$

$$= \int_0^1 x^{a-1}(1-x)^{(1-a)-1}dx = \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)} = \Gamma(a)\Gamma(1-a)$$

<u>Example</u>:- Evaluate $\int_0^{\pi/2}\cos^{2m-1}x\ \sin^{2n-1}x\ dx$

<u>Solution</u> let $\sin x = \sqrt{t}$ ………… (1)



then from the figure $\cos x = \sqrt{1-t}$ ………… (2)

from equation (1) $\cos x\ dx = \frac{1}{2}\ t^{-1/2}\ dt$

$$\Rightarrow \quad dx = \frac{1}{2}\frac{1}{\sqrt{t}}\frac{1}{\cos x}dt \quad \Rightarrow \quad dx = \frac{1}{2}\frac{1}{\sqrt{t}}\frac{1}{\sqrt{1-t}}dt$$

Also from equation when $x = 0$, $t = 0$, and when $x = \frac{\pi}{2}$, $t = 1$

Then $\int_0^{\pi/2}\cos^{2m-1}x\ \sin^{2n-1}x\ dx = \int_0^1(1-t)^{\left(\frac{2m-1}{2}\right)}\ (t)^{\left(\frac{2n-1}{2}\right)}\ \frac{1}{2}\ \frac{1}{\sqrt{t}}\ \frac{1}{\sqrt{1-t}}dt$

Or $= \frac{1}{2}\int_0^1(1-t)^{m-1}(t)^{n-1}dt$

$$\therefore \quad \int_0^{\pi/2}\cos^{2m-1}x\ \sin^{2n-1}x\ dx = \frac{1}{2}\ \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

H.WS

1- Evaluate each of the following

(a) - $\int_0^{\infty}(x+1)^2\, e^{-x^3}\, dx$ (b) - $\int_0^{\infty}\exp(-\sqrt{x})\ dx$ (c) - $\int_0^{\infty}\frac{e^{-x}}{\sqrt{x}}\ dx$

2- Evaluate each of the following

(a) $\int_0^{\pi/2}\sqrt{\cos\theta}\ \ d\theta$ **Answer** $\frac{\sqrt{\pi}}{2}\ \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}$

(b) $\int_0^{\pi/2}\sqrt{\tan\theta}\ \ d\theta$ **Answer** $\frac{1}{2}\ \Gamma\left(\frac{3}{4}\right)\ \Gamma(\frac{1}{4})$

(c) $\int_0^1\frac{dx}{\sqrt{\ln(1/x)}}$ let $\ln(1/x) = z$ **Answer** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(d) $\int_0^1 x^m\left(\ln\frac{1}{x}\right)^n\ dx$ **Answer** $\frac{1}{(m+1)^{n+1}}\ \Gamma(n+1)$

(e) $\int_0^1 \frac{dz}{\sqrt{1-z^4}}$ let $z^4 = x$ **Answer** $\frac{\sqrt{\pi}}{4} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$

3-By setting $2m - 1 = k$ and $n = \frac{1}{2}$ in the result of $\int_0^{\pi/2} \cos^{2m-1} x \ \sin^{2n-1} x \ dx$, show that

$$\int_0^{\pi/2} \cos^k \theta \ d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma[(k+1)/2]}{\Gamma[(k/2)+1]} \quad , \quad k > -1$$

What is $\int_0^{\pi/2} \sin^k \theta \ d\theta$

Partial Fraction Expansion

In many cases the solutions are usually appears as a quotient of polynomials

$$G(x) = Q(x)/P(x) \quad \cdots\cdots\cdots (1)$$

Where Q(x) and P(x) are polynomials of x. It is assumed that the order of P (x) is greater than Q(x). The

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0$$

polynomial P(x) may be written as

…given for the cases of simple pole, multiple – order poles, and complex conjugate poles of G(x)

1- G(x) has simple poles

If all the poles of G(x) are simple and real, equation (1) can be written as

$$G(x) = \frac{Q(x)}{P(x)} = \frac{Q(x)}{(x+x_1)(x+x_2)\cdots\cdots(x+x_n)} \quad \cdots\cdots\cdots (2)$$

where $x_1 \neq x_2 \neq \cdots\cdots \neq x_n$. Applying partial fraction expansion equation (2) becomes to

$$G(x) = \frac{k_1}{(x+x_1)} + \frac{k_2}{(x+x_2)} + \cdots\cdots + \frac{k_n}{(x+x_n)}$$

The coefficients k_i $(i = 1,2,3,\cdots\cdots,n)$ is determined by multiplying both sides of equation (2) by the factor $(x + x_i)$ and then letting x equal to $-x_i$ or

$$k_i = \left[(x + x_i)\frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

Example:- Expand the following by Partial Fraction $G(x) = \frac{5x+3}{x^3+6x^2+11x+6}$

Solution $G(x) = \frac{5x+3}{x^3+6x^2+11x+6} = \frac{5x+3}{(x+1)(x+2)(x+3)}$

then the Partial Fraction form of $G(x)$ is $\frac{5x+3}{(x+1)(x+2)(x+3)} = \frac{k_1}{x+1} + \frac{k_2}{x+2} + \frac{k_3}{x+3}$

to find k_1 multiply both sides by $x + 1$ then let $x = -1$

$$k_1 = \frac{5(-1)+3}{(2-1)(3-1)} = -1$$

$$k_2 = \frac{5(-2)+3}{(1-2)(3-2)} = 7$$

$$k_3 = \frac{5(-3)+3}{(1-3)(2-3)} = -6$$

$$G(x) = \frac{-1}{x+1} + \frac{7}{x+2} - \frac{6}{x+3}$$

2- G(x) has multiple – order poles

If r of the n poles of $G(x)$ are identical, or we say that the pole at $x = -x_i$ is of multiplicity r, $G(x)$ is written as

$$G(x) = \frac{Q(x)}{P(x)} = \frac{Q(x)}{(x+x_1)(x+x_2)\cdots\cdots(x+x_{n-r})(x+x_i)^r} \quad , \quad i \neq 1,2,\cdots\cdots,n-r$$

Then

$$G(x) = \underbrace{\frac{k_1}{(x+x_1)} + \frac{k_2}{(x+x_2)} + \cdots\cdots + \frac{k_{n-r}}{(x+x_{n-r})}}_{n-r \text{ terms of simple poles}} + \underbrace{\frac{A_1}{(x+x_i)} + \frac{A_2}{(x+x_i)^2} + \cdots\cdots + \frac{A_r}{(x+x_i)^r}}_{r-\text{terms of repeated poles}}$$

Where

$$A_r = \left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

$$A_{r-1} = \frac{1}{1!}\frac{d}{dx}\left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

$$A_{r-2} = \frac{1}{2!}\frac{d^2}{dx^2}\left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A_1 = \frac{1}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

Example:- Expand the following function by Partial Fraction $G(x) = \frac{1}{x(x+1)^3(x+2)}$

Solution

$$G(x) = \frac{1}{x(x+1)^3(x+2)} = \frac{k_1}{x} + \frac{k_2}{x+2} + \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3}$$

then $k_1 = \frac{1}{(1^3)(2)} = \frac{1}{2}$

$$k_2 = \frac{1}{(-2)(-2+1)^3} = \frac{1}{2}$$

$$A_3 = [(x+1)^3 G(x)|_{x=-1} = -1$$

$$A_2 = \frac{d}{dx}[(x+1)^3 G(x)|_{x=-1} = \frac{d}{dx}\left[\frac{1}{x(x+2)}\right|_{x=-1} = -\left[\frac{2x+2}{x^2(x+2)^2}\right|_{x=-1} = 0$$

$$A_1 = \frac{1}{2!}\frac{d^2}{dx^2}[(x+1)^3 G(x)\Big|_{x=-1} = \frac{1}{2!}\frac{d^2}{dx^2}\left[\frac{1}{x(x+2)}\right|_{x=-1} = -1$$

Substituting these values $G(x) = \frac{1}{2x} + \frac{1}{2(x+2)} - \frac{1}{x+1} - \frac{1}{(x+1)^3}$

3- G(x) has simple complex – conjugate poles

Suppose that P(x) has simple complex conjugate poles with α_1 as real part and α_2 as imaginary part then

$$G(x) = \frac{Q(x)}{P(x)} = \frac{Q(x)}{(x+\alpha_1-i\alpha_2)(x+\alpha_1+i\alpha_2)}$$

The expansion by partial fraction gives

$$G(x) = \frac{k_{-\alpha_1+i\alpha_2}}{x+\alpha_1-i\alpha_2} + \frac{k_{-\alpha_1-i\alpha_2}}{x+\alpha_1+i\alpha_2}$$

where $k_{-\alpha_1+i\alpha_2} = (x+\alpha_1-i\alpha_2)\, G(x)|_{x=-\alpha_1+i\alpha_2}$

and $k_{-\alpha_1-i\alpha_2} = (x+\alpha_1+i\alpha_2)\, G(x)|_{x=-\alpha_1-i\alpha_2}$

<u>**Example:-**</u> Expand the following function by Partial Fraction $G(x) = \frac{x+2}{(x+1)(x^2+4)}$

<u>Solution</u> $G(x) = \frac{x+2}{(x+1)(x+2i)(x-2i)} = \frac{k_1}{(x+1)} + \frac{k_{-0-2i}}{(x+2i)} + \frac{k_{-0+2i}}{(x-2i)}$

where $k_1 = \frac{x+2}{(x^2+4)}\Big|_{x=-1} = \frac{1}{5}$

$$k_{-0-2i} = \frac{x+2}{(x+1)(x-2i)}\Big|_{x=-0-2i} = \frac{2-2i}{-8-4i}\cdot\frac{-8+4i}{-8+4i} = \frac{24i-8}{80}$$

$$k_{-0+2i} = \frac{x+2}{(x+1)(x+2i)}\Big|_{x=-0+2i} = \frac{2+2i}{-8+4i}\cdot\frac{-8-4i}{-8-4i} = \frac{-24i-8}{80}$$

Roots

Equation $z^n = r^n(\cos n\theta + i\ \sin n\theta)$ can be extended to find the roots of integral orders

Let n-th roots of $z = r(\cos\theta + i\ \sin\theta)$ is defined by number $w = R(\cos\emptyset + i\ \sin\emptyset)$

then, $w^n = z$ or

$$R^n(\cos n\emptyset + i\ \sin n\emptyset) = r(\cos\theta + i\ \sin\theta)$$

comparing the two sides of this equation $R^n = r \quad \Rightarrow \quad R = r^{1/n}$

and the angles of equal complex numbers must either be equal or differ by an integral multiple of 2π

$$n\emptyset = \theta + 2k\pi \quad \text{or} \quad \emptyset = \frac{\theta+2k\pi}{n} \quad \text{where } k = 0, 1, 2, \cdots, n-1$$

$$\boxed{w = z^{1/n} = r^{1/n}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)}$$

Example:- Find the four fourth roots of $-8i$

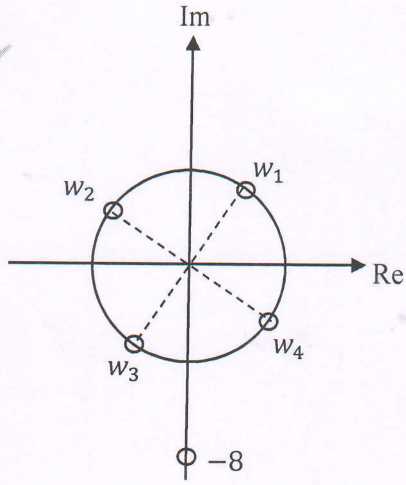
Solution

$$z = 0 - 8i \qquad |z| = \sqrt{0^2 + (-8)^2} = 8$$

$$\theta = 270° = \frac{3\pi}{2} \qquad n = 4$$

Then $R = 8^{1/4}$

$$w = 8^{1/4}\left(\cos\frac{\frac{3\pi}{2}+2k\pi}{4} + i\sin\frac{\frac{3\pi}{2}+2k\pi}{4}\right)$$



$k = 0 \qquad w_1 = 8^{1/4}\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$

$k = 1 \qquad w_2 = 8^{1/4}\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$

$k = 2 \qquad w_3 = 8^{1/4}\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$

$k = 3 \qquad w_4 = 8^{1/4}\left(\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}\right)$

With integral powers and roots defined, the general rational power of complex number can be defined as

$$z^{p/q} = \left(z^{1/q}\right)^p = \left[r^{1/q}\left(\cos\frac{\theta+2k\pi}{q} + i\sin\frac{\theta+2k\pi}{q}\right)\right]^p$$

$$= r^{p/q}\left[\cos\frac{p}{q}(\theta+2k\pi) + i\sin\frac{p}{q}(\theta+2k\pi)\right] \qquad k = 0, 1, 2, \cdots, n-1$$

Example:- Find all the distinct values of $(-1-i)^{4/5}$

Solution

$$z = -1 - i \qquad |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\frac{-1}{-1} = 225^\circ = \frac{5\pi}{4} \qquad q = 5 \qquad p = 4$$

$$z^{4/5} = (-1-i)^{4/5} = r^{4/5}\left[\cos\frac{4}{5}(\theta + 2k\pi) + i\sin\frac{4}{5}(\theta + 2k\pi)\right]$$

$$k = 0 \qquad = 2^{2/5}[\cos\pi + i\sin\pi]$$

$$k = 1 \qquad = 2^{2/5}\left[\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right]$$

$$k = 2 \qquad = 2^{2/5}\left[\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right]$$

$$k = 3 \qquad = 2^{2/5}\left[\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right]$$

$$k = 4 \qquad = 2^{2/5}\left[\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right]$$