Advanced Engineering Analysis

References **1- Advanced Engineering Mathematics by C. RAY WYLIE**

2- Advanced Engineering Mathematics by ERIN KREYS

https://classroom.google.com/c/NzEzMTYzNDM4NTY1?cjc=bznb75o (رمز الصف الإلكتروني)

Definition: D. E. is an eq. that involves one or more derivative, or differential.

Solution of D.E:

Solution of differential equation is a function $f(x)$ that satisfies the D.E.

Example: Show that each function is a solution of the accompanying differential equation:

Note $y' = \frac{dy}{dx} \quad , \quad y'' = \frac{d^2y}{dx^2}$

1 − $xy'' - y' = 0 \quad , \quad y = x^2 + 3$

Solution

$$y' = 2x \quad , \quad y'' = 2$$

Then from D.E $2x - 2x = 0$

2 − $yy'' = 2(y')^2 - 2y' \quad , \quad C_1 y = \tan(C_1 x + C_2)$

Solution

$$y' = \frac{1}{C_1} C_1 \sec^2(C_1 x + C_2) = \sec^2(C_1 x + C_2)$$

$$y'' = 2C_1 \sec(C_1 x + C_2) \cdot \sec(C_1 x + C_2) \tan(C_1 x + C_2)$$

$$y'' = 2C_1 \sec^2(C_1 x + C_2) \cdot \tan(C_1 x + C_2)$$

$$yy'' = 2C_1 \sec^2(C_1 x + C_2) \cdot \left(\frac{1}{C_1} \tan^2(C_1 x + C_2)\right)$$

$$yy'' = 2\sec^2(C_1 x + C_2) \tan^2(C_1 x + C_2)$$

Now, $2(y')^2 - 2y' = 2(\sec^2(C_1 x + C_2))^2 - 2\sec^2(C_1 x + C_2)$

$$= 2\sec^2(C_1 x + C_2)\,(\sec^2(C_1 x + C_2) - 1)$$

$$= 2\sec^2(C_1 x + C_2) \tan^2(C_1 x + C_2)$$

H.W

$$2y' + 3y = e^{-x} \quad , \quad y = e^{-x} + Ce^{-(3/2)x}$$

The General Linear Second – Order Differential Equation

The general linear ordinary differential equation of the second order can be written in the standard form

$$a(x)y'' + b(x)y' + c(x)y = f(x) \qquad \ldots\ldots\ldots\ldots (1)$$

Where $y'' = \frac{d^2y}{dx^2}$ and $y' = \frac{dy}{dx}$ y = dependent variable, and x = independent variable

The $a(x),\ b(x)\ and\ c(x)$ are coefficient. Equation (1) is said to be **nonhomogeneous.** If $f(x)$ is identically zero, we have the so-called **homogeneous** equation.

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad \textbf{(homogeneous)}$$

The Homogeneous Linear Second – Order Differential Equation with Constant Coefficients

When $a(x),\ b(x)\ and\ c(x)$ are constants the general linear second order differential equation can be written in the standard form

$$ay'' + by' + cy = 0 \qquad \ldots\ldots\ldots\ldots (2)$$

Let the solution of equation (2) be in the form

$y = e^{mx}$ where m is a constant to be determined.

Substituting $y' = me^{mx}$ and $y'' = m^2e^{mx}$ in to equation (2)

$$e^{mx}(am^2 + bm + c) = 0$$

Since $e^{mx} \neq 0$ then $\boxed{am^2 + bm + c = 0}$ $\Rightarrow$ **(characteristic or auxiliary equation)**

The roots of characteristic equation can be obtained by

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, if $b^2 - 4ac = 0$ then $m_1 = m_2$ Real and equal Complete solution of the equation (2) is

$$\boxed{y(x) = c_1e^{m_1x} + c_2xe^{m_1x}}$$

if $b^2 - 4ac > 0$ then $m_1 \neq m_2$ Real and unequal Complete solution of the equation (2) is

$$\boxed{y(x) = c_1e^{m_1x} + c_2e^{m_2x}}$$

and if

$b^2 - 4ac < 0$ then $m_1 and\ m_2$ are complex conjugate $m_1 = p \mp iq$
Complete solution of the equation (2) is

$$\boxed{y(x) = e^{px}(A\cos qx + B\sin qx)}$$

Example Find complete solution of the equation $y'' + 7y' + 12y = 0$

Solution

let $y = e^{mx}$ then from given equation $e^{mx}(m^2 + 7m + 12) = 0$

then the characteristic equation becomes to $m^2 + 7m + 12 = 0$

$a = 1$ $b = 7$ $c = 12$

And its roots are $m_1 = -3$ $m_2 = -4$

Since $m_1 \neq m_2$ Real and unequal, a complete solution is

$$y(x) = c_1e^{-3x} + c_2e^{-4x}$$

Example Find complete solution of the equation $y'' + 2y' + 5y = 0$

Solution

let $y = e^{mx}$ then from given equation $e^{mx}(m^2 + 2m + 5) = 0$

the characteristic equation in this case is $m^2 + 2m + 5 = 0$

$a = 1$ $b = 2$ $c = 5$

And its roots are $m_1 = -1 + 2i$ $m_2 = -1 - 2i$

Since $m_1 and\ m_2$ are complex conjugate it clear that $p = -1$ $q = 2$

Then, a complete solution of equation is

$$y(x) = e^{-1x}(A\cos 2x + B\sin 2x)$$

Operator Notation

By definition $y' = \frac{dy}{dx} = Dy$

Then by repetition of the process of differentiation

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2y$$

Similarly $$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = D(D^2y) = D^3y$$

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The operator D can be handled in many respects as though it were a simple algebraic quantity.

Particular Solution

In particular solutions, the two arbitrary constants in the complete solution must usually be determined to fit given initial (or boundary) conditions on y and y'.

Example Find the solution of the equation $y'' - 4y' + 4y = 0$ for which $y = 3$ and $y' = 4$ when $x = 0$

Solution

Using operator notation, the given $D^2y - 4Dy + 4y = 0 \Rightarrow \quad (D^2 - 4D + 4)y = 0$

Replace D by m the characteristic equation of given equation becomes to $m^2 - 4m + 4 = 0$ and its roots are $m_1 = m_2 = 2$; hence the complete solution is

$$y(x) = c_1e^{2x} + c_2xe^{2x}$$

Now to find the particular solution, differentiate $y(x)$ with respect to x

$$y'(x) = 2c_1e^{2x} + c_2e^{2x} + 2c_2xe^{2x} = (2c_1 + c_{2)}e^{2x} + 2c_2xe^{2x} =$$

The substituting the given conditions into the equations for y and y', respectively, we have

$$3 = c_1 \quad \text{and} \quad 4 = 2c_1 + 4c_2$$

Hence, $c_1 = 3$, and $c_2 = -2$

Then the required solution is $y(x) = 3e^{2x} - 2xe^{2x}$

H.WS

1- Find a complete solution of each of the following equations:

a- $y'' + 5y' = 0$ **Answer** $y(x) = c_1 + c_2e^{-5x}$

b- $(9D^2 - 12D + 4)y = 0$ **Answer** $y(x) = c_1e^{2x/3} + c_2xe^{2x/3}$

c- $y'' + 10y' + 26y = 0$ **Answer** $y(x) = e^{-5x}(A\cos x + B\sin x)$

2- Find a particular solution of each of the following equations which satisfies the given conditions: