Constant Coefficient Homogeneous Equation of Higher Order

Consider the following homogeneous, liner, constant – coefficient equation of higher order

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

where n is order of derivative and $n > 2$

the substitution $y = e^{mx}$ leads to the characteristic equation

$$a_0 m^n + a_1 m^n + \cdots + a_{n-1} m + a_n = 0 \qquad \text{(n- roots equation)}$$

Example Find a complete solution of the equation $y''' + 3y'' + 3y' + y = 0$

Solution

In this case the characteristic equation is $m^3 + 3m^2 + 3m + 1 = 0$

One roots of this equation is $m = -1$

then to find the other two roots multiply the characteristic equation by $\frac{m+1}{m+1}$ the equation becomes to

$$(m+1)\frac{(m^3+3m^2+3m+1)}{m+1} = (m+1)(m+1)^2 = (m+1)^3$$

$\therefore$ the roots of characteristic equation are $m_1 = m_2 = m_3 = -1$

the complete solution is $y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$

Example Find a complete solution of the equation $y''' + 5y'' + 9y' + 5y = 0$

Solution

To find the characteristic equation $(D^3 + 5D^2 + 9D + 5)y = 0 \quad \Rightarrow \quad m^3 + 5m^2 + 9m + 5 = 0$

by inspection one roots of this equation is $= -1$, then to find the other two roots multiply the characteristic equation by $\frac{m+1}{m+1}$ the equation becomes to

$$(m+1)\frac{(m^3+5m^2+9m+5)}{m+1} = 0 \quad \Rightarrow \quad (m+1)(m^2+4m+5) = 0$$

$\therefore$ the roots of characteristic equation are $m_1 = -1$, $m_{2,3} = -2 \pm i$

so the total homogeneous solution is $c_1 e^{-x} + e^{-2x}(c_2 cosx + c_3 sinx)$

Example For what nonzero values of λ, if any, does the equation $y^{iv} - \lambda^4 y = 0$ have solutions which satisfy the four conditions $y(0) = y''(0) = y(l) = y''(l) = 0$ and are not identically zero? What are these solutions if they exist ?

Solution

The characteristic equation in this case is $m^4 - \lambda^4 = 0$ to find the roots of this equation let $m^2 = q$ then $q^2 - \lambda^4 = 0$ $\Rightarrow$ $q = \mp\lambda^2$ then $m = \mp\lambda, \mp i\lambda$

Hence, a complete solution is $y(x) = A\cos\lambda x + B\sin\lambda x + c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

or $y(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + E\sinh\lambda x$

differentiation this twice gives us

$$y''(x) = \lambda^2(-A\cos\lambda x - B\sin\lambda x + C\cosh\lambda x + E\sinh\lambda x)$$

From the first two conditions, we obtain the relations

$$A + C = 0$$

$$\lambda^2(-A + C) = 0 \quad \text{since } \lambda \neq 0 \quad \Rightarrow \quad A = C = 0$$

From the two last conditions, we obtain the relations

$$B\sin\lambda l + E\sinh\lambda l = 0$$

$$\lambda^2(-B\sin\lambda l + E\sinh\lambda l) = 0$$

Dividing both sides by λ^2 and adding these equations, we find

$$2E\sinh\lambda l = 0 \qquad \sinh\lambda l \neq 0 \qquad \therefore\ E = 0$$

which implies that $B\sin\lambda l = 0$, now if $B = 0$ the solution would be identically zero, thus we must have $\sin\lambda l = 0$ or $\lambda l = n\pi$ $\Rightarrow$ $\lambda = \frac{n\pi}{l}$

then by substitution this value of λ $y_n = B_n \sin(n\pi x/l)$

Variable Coefficients Differential Equation

$a_0x^2y'' + a_1xy' + a_2y = f(x)$ **Euler – Cauchy**

The Euler – Cauchy Differential Equation

There are one type of linear equation with variable coefficients, by simple change of independent variable it can always be transformed into linear equation with constant coefficients.

The equation

$a_0x^2y'' + a_1xy' + a_2y = f(x)$ is called **Euler – Cauchy** equation

Note that The coefficients of each derivative is proportional to the corresponding power of the independent variable.

Then by changing the independent variable from x to z by means of the substitution

$x = e^z$ or $z = \ln x$ so $y = f(z)$ $\Rightarrow$ $\frac{dz}{dx} = \frac{1}{x}$

Note that $z = f(x)$ and $y = f(z)$

Then $$y' = \frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}.\frac{dy}{dz} \quad \cdots\cdots\cdots (a)$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \quad \cdots\cdots\cdots (b)$$

$$y''' = \frac{d^3y}{dx^3} = \frac{2}{x^3}\frac{dy}{dz} - \frac{3}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^3}\frac{d^3y}{dz^3} \quad \cdots\cdots\cdots (c)$$

Example:-Solve $x^2y'' - 4xy' + 6y = 0$

Solution

from equations (a) and (b), the given D.E can be transformed

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right) - 4x\left(\frac{1}{x}.\frac{dy}{dz}\right) + 6y = 0$$

$$\frac{d^2y}{dz^2} - 5\frac{dy}{dz} + 6y = 0 \qquad \text{(constant - coefficient)}$$

the characteristic equation $\Rightarrow$ $m^2 - 5m + 6 = 0$

$\therefore$ the roots of characteristic equation are $m_1 = 2$, $m_2 = 3$

then the general solution is $y(z) = c_1e^{2z} + c_2e^{3z}$

finally transform from z- domain to x –domain by substituting $z = \ln x$

$$y(x) = c_1\, e^{2(\ln x)} + c_2\, e^{3(\ln x)} \quad \Rightarrow \quad y(x) = c_1\, x^2 + c_2\, x^3$$

H.Ws
Solve

1- $x^2y'' - 3xy' + 4y = 0$ Answer $y(x) = (c_1 + c_2 \ln x)x^2$

2- $x^2y'' - xy' + y = x^5$ Answer $y(x) = c_1 + c_2 x \ln x + \frac{x^5}{16} +$

3- $2x^2y'' + 5xy' + y = 3x + 2$ Answer $y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2}{x} + 2 + \frac{x}{2}$

Euler- Cauchy Equation of Higher Order

Example Solve the nonhomogeneous Euler- Cauchy equation $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 lnx$

Solution

From equations (a) and (b), the given D.E can be transformed the homogeneous part to

$$y''' - 6y'' + 11y' - 6y = 0 \quad \text{where} \quad y = f(z)$$

then the characteristic equation is $\Rightarrow m^3 - 6m^2 + 11m - 6 = 0$

$\therefore$ the roots of characteristic equation are $m_1 = 1$, $m_2 = 2,$ $m_3 = 3$

$y_1(z) = e^z$ $y_2(z) = e^{2z}$ $y_3(z) = e^{3z}$

then $y_1(x) = x$ $y_2(x) = x^2$ $y_3(x) = x^3$

then the complete solution is $y_h(x) = c_1 x + c_2 x^2 + c_3 x^3$

now to find the particular solution we use the method of variation of parameter where

$$y_p = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x) dx$$ where n = order of differential equation

For n = 3 $y_p = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)dx + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)dx + y_3(x) \int \frac{W_3(x)}{W(x)} r(x)dx$

where $y_1(x) = x$ $y_2(x) = x^2$ $y_3(x) = x^3$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Recall that $r(x) = \frac{x^4 lnx}{x^3} = x\, lnx$ (by comparison the given differential equation by standard form)

$W_1/W = x/2$ $\quad W_2/W = -1$ $\quad W_3/W = 1/(2x)$

$$y_p = x\int \frac{x}{2} x\, lnx\, dx - x^2 \int x\, lnx\, dx + x^3 \int \frac{1}{2x} x\, lnx dx$$

Now $\int \ln x\, dx$, let $u = \ln x$, $du = \frac{dx}{x}$

$dv = dx$, $v = x$

$$\int \ln x\, dx = x \ln x - \int x \frac{dx}{x}$$

$$\therefore \boxed{\int \ln x\, dx = x \ln x - x}$$

$\int x \ln x\, dx$ let $u = \ln x$, $du = \frac{dx}{x}$

$dv = xdx$, $v = \frac{x^2}{2}$

$$\int x \ln x\, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{dx}{x}$$

$$\therefore \boxed{\int x \ln x\, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}}$$

$\int x^2 \ln x\; dx = \int x\, x \ln x\, dx$, $u = x$, $du = dx$

$dv = x \ln x\, dx$, $v = \frac{x^2}{2} \ln x - \frac{x^2}{4}$

$$\int x^2 \ln x\; dx = \frac{x^3}{2} \ln x - \frac{x^3}{4} - \int \left(\frac{x^2}{2} \ln x - \frac{x^2}{4}\right) dx$$

or $\frac{3}{2}\int x^2 \ln x\; dx = \frac{x^3}{2} \ln x - \frac{x^3}{4} + \frac{x^3}{12}$ $\quad \therefore \boxed{\int x^2 \ln x\; dx = \frac{x^3}{3} \ln x - \frac{x^3}{9}}$

$$y_p = \frac{x}{2}\left(\frac{x^3}{3} \ln x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2} \ln x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x \ln x - x)$$

$$y_p = \frac{1}{6} x^4 \left(\ln x - \frac{11}{6}\right)$$

$$y(x) = c_1 x + c_2 x^2 + c_3 x^3 + \frac{1}{6} x^4 \left(\ln x - \frac{11}{6}\right)$$