Partial Differential Equations (PDE)

When the differential equation consists of dependent variable and more than one independent variable the differential equation becomes to partial differential equations. The general form of partial differential equation is

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = g(u, u_x, u_y, x, y)\,\frac{\partial^2 u}{\partial x^2} \qquad u_{yy} = \frac{\partial^2 u}{\partial x^2}$$

The equation (1) is said to be **hyperbolic**, **parabolic**, or **elliptic** throughout a region R according as $B^2(x,y) - A(x,y)C(x,y)$ is greater than, equal to, or less than zero at all points of R.

The simplest, and in elementary applications the most important, examples of hyperbolic, parabolic, and elliptic partial differential equations are, respectively,

(a)- The wave equation $a^2\,\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad \Longrightarrow \quad B^2 - AC > 0$

(b)- The heat equation $\frac{\partial^2 u}{\partial x^2} - a^2\,\frac{\partial u}{\partial t} = 0 \quad \Longrightarrow \quad B^2 - AC = 0$

(c)- Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Longrightarrow \quad B^2 - AC < 0$

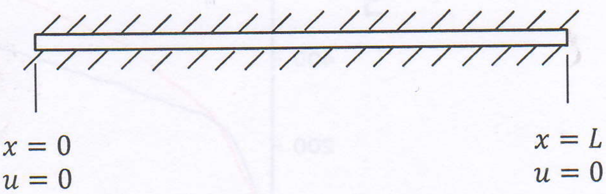
Solution of Partial Differential Equations

Example:- A rod of length L is perfectly insulated against the flow of heat. The rod, which is so thin that heat flow in it can be assumed to be one-dimensional, is initially at uniform temperature $u = 100\ \mathrm{C}^\circ$. Find the temperature $u(x,t)$ at any point in the rod at any subsequent time, if at $t = 0$ the temperature at each end of the rod is suddenly dropped to 0 C° and maintained at that temperature thereafter.

Solution

The governing equation of this problem is

$$\frac{\partial^2 u}{\partial x^2} - a^2\,\frac{\partial u}{\partial t} = 0 \qquad \Longrightarrow \qquad \frac{\partial^2 u}{\partial x^2} = a^2\,\frac{\partial u}{\partial t}$$



subjected to the boundary conditions

$$u(0,t) = 0$$

$$u(L,t) = 0$$

and the initial condition $u(x,0) = f(x) = 100$

we assume that the solution for the temperature $u(x,t)$ exist as products of a function of x alone and a function of t alone

$$u(x,t) = X(x)T(t)$$

If this is the case

$$\frac{\partial^2 u}{\partial x^2} = X''\,T \qquad \text{and} \qquad \frac{\partial u}{\partial t} = X\dot{T}$$

Substituting these into the heat equation, we obtain

$$X''\,T = a^2 X\dot{T}$$

Dividing by product XT then gives

$\frac{X''}{X} = a^2\frac{\dot{T}}{T} = \text{constant} = \mu$ (since the left hand is a function of x and right hand is a function of t)

Thus the determination of solutions of the original partial differential equation has been reduced to the determination of solutions of the **two ordinary differential** equations

$$X'' = \mu X \qquad \cdots\cdots\cdots (1) \quad \text{and} \qquad \dot{T} = \frac{\mu}{a^2}T \qquad \cdots\cdots\cdots (2)$$

Assuming that we need consider only real values of μ, there are three cases to investigate:

$$\mu > 0 \qquad \mu = 0 \qquad \mu < 0$$

Let $\mu > 0$, say $\mu = \lambda^2$, where $\lambda > 0$, then the differential equations (1) and (2) and their solutions are

$$X'' = \lambda^2 X \qquad \dot{T} = \frac{\lambda^2}{a^2}T$$

$$X(x) = A\cosh\lambda x + B\sinh\lambda x \quad , \qquad T(t) = Ce^{(\lambda^2/a^2)t}$$

$$u(x,t) = X(x)T(t) = (A\cosh\lambda x + B\sinh\lambda x)\left(Ce^{(\lambda^2/a^2)t}\right)$$

This must be rejected immediately because $u \longrightarrow \infty$ when $t \longrightarrow \infty$

Let $\mu = 0$, then the differential equations (1) and (2) and their solutions are

$$X'' = 0 \qquad \dot{T} = 0$$

$$X(x) = Ax + B \qquad T(t) = C$$

then $u(x,t) = X(x)T(t) = Ax + B$ [The coefficient C is absorbed in the arbitrary constants A and B]

applying boundary conditions $u(0,t) = u(L,t) = 0 \Longrightarrow \quad A = B = 0$

Let $\mu < 0$, say $\mu = -\lambda^2$, where $\lambda > 0$, then the differential equations (1) and (2) and their solutions are

$$X'' = -\lambda^2 X \qquad \dot{T} = -\frac{\lambda^2}{a^2}T$$

$X(x) = A\cos\lambda x + B\ \sin\lambda x \qquad , \qquad T(t) = Ce^{(-\lambda^2/a^2)t}$

$$u(x,t) = X(x)T(t) = (A\cos\lambda x + B\ \sin\lambda x)e^{(-\lambda^2/a^2)t}$$

Now, apply the boundary condition $u(0,t) = 0 \quad \Longrightarrow \quad A = 0 \qquad$ since $e^{(-\lambda^2/a^2)t} \neq 0$

and $\quad u(L,t) = 0 \quad \Longrightarrow \quad B\ \sin\lambda L = 0\ \ B \neq 0 \qquad \sin\lambda L = 0$

$\lambda L = n\pi \qquad \lambda = \frac{n\pi}{L} \qquad n = 1,2,3,4,\cdots$

$$U_n(x,t) = B_n \sin\frac{n\pi}{L}x\ \ e^{(-n^2\pi^2/L^2a^2)t}$$

So $\quad u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\ \ e^{(-n^2\pi^2/L^2a^2)t}$

This equation is satisfy only boundary conditions but not initial condition, now our aim is to find the constant B_n where $u(x,t)$ satisfy both boundary and initial condition

applying initial condition $\qquad u(x,0) = f(x) = 100$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$$

Then

$$B_n = \frac{\int_0^L f(x)\ \sin(n\pi/L)xdx}{\int_0^L \sin^2(n\pi/L)xdx} = \frac{\int_0^L 100\ \sin(n\pi/L)xdx}{\int_0^L \frac{1}{2}[1-\cos(n\pi/L)xdx]} = \frac{100\int_0^L \sin(n\pi/L)xdx}{L/2}$$

$$B_n = \frac{200}{L}\left[-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L = \frac{200}{n\pi}[1-\cos n\pi]$$

But $\quad [1-\cos n\pi] = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{400}{n\pi}\sin\frac{n\pi}{L}x\ \ e^{(-n^2\pi^2/L^2a^2)t} \qquad \text{where } n \text{ is odd}$$

Example:- A sheet of metal coincides with the square in the xy-plane whose vertices are the points (0,0) , (1,0) , (1,1) and (0,1). The two faces of the sheet are perfectly insulated and the sheet is so thin that heat flow in it can be regarded as two-dimensional. The edges parallel to the x axis are insulated, and the left-hand edge is maintained at the constant temperature 0. If the temperature distribution $u(1,y) = f(y)$ is maintained along the right-hand edge, find the steady-state temperature distribution throughout the sheet.

Solution

The governing equation of this problem is two – dimensional heat equation

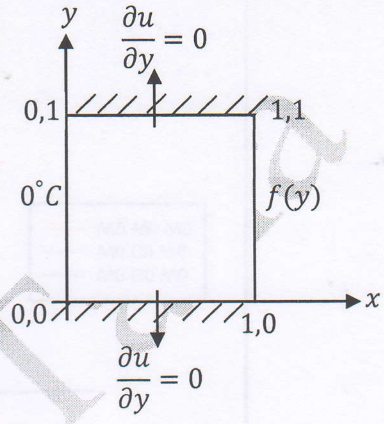
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - a^2\frac{\partial u}{\partial t} = 0$$

Since we are asked to find steady-state temperature distribution in the sheet $\Longrightarrow$ $\frac{\partial u}{\partial t} = 0$

Under these assumptions, the two – dimensional heat equation reduced to Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{or} \qquad \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

subjected to the following boundary conditions



$$u(0, y) = 0$$
$$u(1, y) = f(y)$$
$$\left.\frac{\partial u}{\partial y}\right|_{x,0} = 0$$
$$\left.\frac{\partial u}{\partial y}\right|_{x,1} = 0$$

we assume that the solution for the temperature $u(x, y)$ exist as products of a function of x alone and a function of y alone

$$u(x, y) = X(x)Y(y)$$

If this is the case

$$\frac{\partial^2 u}{\partial x^2} = X''\,Y \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = XY'' \qquad \text{where} \qquad X'' = \frac{d^2X}{dx^2} \quad \text{and} \quad Y'' = \frac{d^2Y}{dy^2}$$

Substituting these into the two – dimensional heat equation, we obtain

$$X''\,Y = -XY''$$

Dividing by product XY then gives

$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = \mu$ (since the left hand is a function of x and right hand is a function of y)

Thus the determination of solutions of the original partial differential equation has been reduced to the determination of solutions of the **two ordinary differential** equations

$$X'' = \mu X \quad \cdots\cdots\cdots\cdots (1) \qquad \text{and} \qquad Y'' = -\mu Y \quad \cdots\cdots\cdots\cdots (2)$$

Now, let $\mu < 0$, say $\mu = -\lambda^2$, where $\lambda > 0$, then the differential equations (1) and (2) and their solutions are

$$X'' = -\lambda^2 X \qquad\qquad Y'' = \lambda^2 Y$$

$$X(x) = A\cos\lambda x + B\,\sin\lambda x \qquad\qquad Y(y) = C\cosh\lambda y + D\,\sinh\lambda y$$

then

$$u(x,y) = X(x)Y(y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y)$$

Now, from $u(0,y) = 0$ $\quad u(0,y) = 0 = A(C\cosh\lambda y + D\sinh\lambda y)$

since $0 \le y \le 1 \quad \Longrightarrow \quad A = 0$

absorbing the B coefficient in C and D

$$u(x,y) = \sin\lambda x\,(C\cosh\lambda y + D\sinh\lambda y)$$

we must now impose the boundary conditions that hold along the upper and lower edges of the sheet [every point on each of these edges the normal temperature gradient must be zero], or

$$\frac{\partial u}{\partial y} = \sin\lambda x\,(\lambda C\sinh\lambda y + \lambda D\cosh\lambda y)$$

For $y = 0$ (lower edge)

$$0 = \lambda D\sin\lambda x \qquad \cdots\cdots\cdots (3)$$

For $y = 1$ (upper edge)

$$\sin\lambda x\,(\lambda C\sinh\lambda + \lambda D\cosh\lambda) = 0 \cdots\cdots\cdots (4)$$

Since these conditions must hold for all $0 \le x \le 1$ then, from (4) $D = 0$

From (4) $\quad \lambda C\sinh\lambda = 0 \quad \sinh\lambda \neq 0 \quad C = 0$

$\therefore \quad \mu < 0$ not satisfy

Try $\mu = 0 \quad \Longrightarrow \quad X'' = 0 \qquad Y'' = 0$

$X(x) = Ax + B \qquad Y(y) = Cy + D$

$$u(x,y) = (Ax + B)(Cy + D)$$

Applying the boundary condition $u(0,y) = 0 \qquad 0 = B(Cy + D)$

Then either $B = 0$ or C and D both is zero

To obtain nontrivial solution ,let $B = 0$

$u(x,y) = x(Cy + D)$ [The coefficient A is absorbed in the arbitrary constants C and D]

To impose the insulated conditions which hold along the upper and lower edges

$$\frac{\partial u}{\partial y} = Cx$$

This will be zero at $y = 0$ and at $y = 1$, if and only if $C = 0$

$u(x,y) = Dx$ for convenience let $D = \frac{1}{2}C_o$ then $\boxed{u(x,y) = \frac{1}{2}C_o\,x}$

Now, , let $\mu > 0$, say $\mu = \lambda^2$, where $\lambda > 0$, then the differential equations (1) and (2) and their solutions are

$X'' = \lambda^2 X \qquad Y'' = -\lambda^2 Y$

$$X(x) = A\cosh\lambda x + B\sinh\lambda x \qquad Y(y) = C\cos\lambda y + D\sin\lambda y$$

then

$$u(x,y) = X(x)Y(y) = (A\cosh\lambda x + B\sinh\lambda x)(C\cos\lambda y + D\sin\lambda y)$$

Applying the boundary condition $u(0,y) = 0$

$$0 = A(C\cos\lambda y + D\sin\lambda y) \quad \Rightarrow \quad A = 0$$

$$u(x,y) = \sinh\lambda x\,(C\cos\lambda y + D\sin\lambda y)$$

For the upper and lower edges $\frac{\partial u}{\partial y} = \sinh\lambda x\,(-C\lambda\sin\lambda y + D\lambda\cos\lambda y)$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \qquad 0 = D\lambda\sinh\lambda x \quad \Rightarrow \quad D = 0$$

so $\frac{\partial u}{\partial y} = -C\lambda\sin\lambda y\sinh\lambda x$

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} = 0 \quad , \qquad 0 = -C\lambda\sin\lambda\ \sinh\lambda x \quad \Rightarrow \quad C \neq 0 \qquad \sin\lambda = 0$$

$$\lambda = n\pi$$

Then $u_n(x,y) = C_n\sinh n\pi x\cos n\pi y$

And the solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is

$$u(x,y) = \tfrac{1}{2}C_o x + \sum_{n=1}^{\infty} C_n\sinh n\pi x\cos n\pi y$$

This equation satisfies

1- Laplace equation
2- The boundary condition $u(0,y) = 0$
3- The boundary condition $\left.\frac{\partial u}{\partial y}\right|_{x,0} = 0$
4- The boundary condition $\left.\frac{\partial u}{\partial y}\right|_{x,1} = 0$

The final boundary condition, namely, that along the right-hand edge of the sheet the temperature distribution $u(1,y) = f(y)$ is maintained. Let $x = 1$

$$u(1,y) = f(y) = \tfrac{1}{2}C_o + \sum_{n=1}^{\infty}(C_n\sinh n\pi)\cos n\pi y$$

To determine the constants C_o and C_n multiply both sides by $\cos n\pi y$ and integrate from 0 to 1

$$C_o = 2\int_0^1 f(y)\,dy \qquad \text{and} \qquad C_n = \frac{1}{\sinh n\pi}\frac{\int_0^1 f(y)\cos n\pi y\,dy}{\int_0^1 \cos^2 n\pi y\,dy}$$

or