Power Series Method of Solving Differential Equation

Power series is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m x^m = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Where $a_o, a_1, a_2, a_3, \cdots$ are constant called coefficients.

Familiar examples of power series are the Maclurian series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \cdots \qquad \text{(Geometric series} \quad |x| \le 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^m}{m!} + \cdots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \cdots \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

To solve the differential equation we assume a solution in the form of a power series with unknown coefficients

$$y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad \cdots\cdots\cdots\cdots (1)$$

and inserting this series and the series obtained by termwise differentiation

$$y'(x) = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \qquad \cdots\cdots\cdots\cdots (2)$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3 \times 2\, a_3 x + 4 \times 3\, a_4 x^2 \cdots \qquad \cdots\cdots\cdots\cdots (3)$$

Example Solve the following ordinary differential equation by power series $y' = 2xy$

Solution by inserting (1) and (2) into the given equation

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

Multiplying $2x$ inside the bracket we obtain

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + \cdots$$

For this equation to hold the two coefficients of every power of x on both sides must be equal, that is

$$a_1 = 0, \quad 2a_2 = 2a_0, \quad 3a_3 = 2a_1, \quad 4a_4 = 2a_2, \quad 5a_5 = 2a_3, \quad 6a_6 = 2a_4,$$

$$\Rightarrow a_3 = a_5 = a_{\text{odd}} = 0$$

and $\quad a_2 = a_0, \quad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \quad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \quad \cdots$

$$y(x) = a_0 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) = a_0 e^{x^2} \qquad \text{where } a_0 \text{ is arbitrary}$$

Shifting Index Method

By substituting equation (1) and (2) into the given equation

$$1 \cdot a_1 x^0 + \sum_{m=2}^{\infty} m a_m x^{m-1} = 2x \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} 2a_m x^{m+1}$$

Let $m = s + 2$ in the left then the summation, which started with $m = 2$, now starts with $s = 0$. On the right we simply make a change of notation $m = s$.

$$a_1 + \sum_{s=0}^{\infty} (s+2) a_{s+2} x^{s+1} = \sum_{s=0}^{\infty} 2a_s x^{s+1}$$

nce,

$a_1 = 0$, $(s+2)a_{s+2} = 2a_s$ or $a_{s+2} = \frac{2}{s+2}a_s$

Then, for $s = 0,1,2,\cdots$ we have $a_2 = \frac{2}{2}a_0,$ $a_3 = \frac{2}{2}a_1 = 0,$ $a_4 = \frac{1}{2}a_2$

Example Solve the following ordinary differential equation by power series $y'' + y = 0$

Solution by inserting (1) and (3) into the given equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

To obtain the same general power on both sides, we set $m = s+2$ in the first series and $m = s$ in the second, this gives

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}\, x^s = -\sum_{s=0}^{\infty} a_s x^s$$

Each power x^s must have the same coefficient on both sides. Hence

$$(s+2)(s+1)a_{s+2} = -a_s \Rightarrow \quad a_{s+2} = \frac{-a_s}{(s+2)(s+1)} \quad ,(s = 0,1,2,\cdots)$$

$$a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!} \quad , \quad a_3 = -\frac{a_1}{3\cdot 2} = -\frac{a_1}{3!}$$

$$a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!} \quad , \quad a_5 = -\frac{a_3}{5\cdot 4} = -\frac{a_1}{5!}$$

Where a_0 and a_1 remains arbitrary. With these coefficients the series (1) becomes

$$y(x) = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$y(x) = a_0 \cos x + a_1 \sin x$$

H.W

Solve by power series

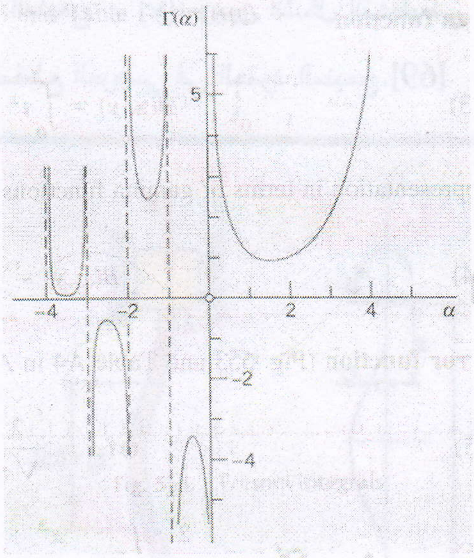
1- $(x+1)y' - (x+2)y = 0$ Answer $y(x) = a_0(1+x)e^x$

2- $xy' - (x+2)y = -2x^2 - 2x$ Answer $y(x) = a_2 x^2 e^x + 2x$

Gamma Function

The Gamma function $\Gamma(\alpha)$ is defined by the integral

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}\, t^{\alpha-1} dt \qquad \alpha > 0$$



Integrating by part let $u = e^{-t} \quad \Rightarrow \quad du = -e^{-t}dt$

$$dv = t^{\alpha-1}dt \quad \Rightarrow \quad v = \frac{t^{\alpha}}{\alpha}$$

$$\int u\,dv = uv - \int v\,du \quad \Rightarrow \quad \therefore\ \Gamma(\alpha) = e^{-t}\frac{t^{\alpha}}{\alpha}\Big]_0^{\infty} + \int_0^{\infty} \frac{t^{\alpha}}{\alpha} e^{-t}dt$$

But $e^{-t}\frac{t^{\alpha}}{\alpha}\Big]_0^{\infty} = 0$ then $\Gamma(\alpha) = \frac{1}{\alpha}\int_0^{\infty} t^{\alpha+1-1}e^{-t}dt = \frac{\Gamma(\alpha+1)}{\alpha}$

PROOF

Since $e^{-t}t^{\alpha} = \frac{t^{\alpha}}{e^{t}}$ using L'opital rule $\lim_{t\to\infty}\frac{\alpha t^{\alpha-1}}{e^{t}} = \lim_{t\to\infty}\frac{\alpha(\alpha-1)t^{\alpha-2}}{e^{t}} = \cdots\cdots =$
$\lim_{t\to\infty}\frac{\alpha(\alpha-1)(\alpha-2)\cdots\cdots(\alpha-k+1)t^{\alpha-k}}{e^{t}} = 0$ for $k > \alpha$

now integrating by part twice and more

$$\boxed{\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha} = \frac{\Gamma(\alpha+2)}{\alpha(\alpha+1)} = \cdots\cdots\cdots\cdots = \frac{\Gamma(\alpha+k+1)}{\alpha(\alpha+1)(\alpha+2)\cdots\cdots(\alpha+k)}} \qquad \cdots\cdots\cdots\ (*)$$

Where k = times of integration -1

Note that $\Gamma(\alpha) \to \infty$ when $(\alpha) = 0, -1, -2, -3, -4, \cdots$

when $\propto = 1$, $\Gamma(1) = \int_0^\infty e^{-t}\, t^{1-1} dt = -e^{-t}\big]_0^\infty = -\frac{1}{e^t}\Big]_0^\infty = 1$

Now from $\Gamma(\propto + 1) = \propto \Gamma(\propto)$ let $\propto = 1$ $\Gamma(2) = 1\Gamma(1) = 1!$

let $\propto = 2$ $\Gamma(3) = 2\Gamma(2) = 2!$

let $\propto = 3$ $\Gamma(4) = 3\Gamma(3) = 3!$

let $\propto = 4$ $\Gamma(5) = 4\Gamma(4) = 4!$

$\vdots$ $\vdots$

let $\propto = k$ $\Gamma(k+1) = k!$

So the Gamma function may be regarded as generalization of the factorial function.

Table A2 Gamma Function [see (24) in App. A3.1]

α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$
1.00	1.000 000	1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384
1.02	0.988 844	1.22	0.913 106	1.42	0.886 356	1.62	0.895 924	1.82	0.936 845
1.04	0.978 438	1.24	0.908 521	1.44	0.885 805	1.64	0.898 642	1.84	0.942 612
1.06	0.968 744	1.26	0.904 397	1.46	0.885 604	1.66	0.901 668	1.86	0.948 687
1.08	0.959 725	1.28	0.900 718	1.48	0.885 747	1.68	0.905 001	1.88	0.955 071
1.10	0.951 351	1.30	0.897 471	1.50	0.886 227	1.70	0.908 639	1.90	0.961 766
1.12	0.943 590	1.32	0.894 640	1.52	0.887 039	1.72	0.912 581	1.92	0.968 774
1.14	0.936 416	1.34	0.892 216	1.54	0.888 178	1.74	0.916 826	1.94	0.976 099
1.16	0.929 803	1.36	0.890 185	1.56	0.889 639	1.76	0.921 375	1.96	0.983 743
1.18	0.923 728	1.38	0.888 537	1.58	0.891 420	1.78	0.926 227	1.98	0.991 708
1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384	2.00	1.000 000

<u>Example:-</u> Evaluate $\int_0^\infty e^{-t} t^{1/2} dt$

<u>Solution</u> since $\frac{1}{2} = \frac{1}{2} + 1 - 1 = 1.5 - 1$ then $\int_0^\infty e^{-t} t^{1/2} dt = \int_0^\infty e^{-t} t^{1.5-1} dt =$ $\Gamma(1.5) = 0.886227$

<u>Example:-</u> Evaluate $\int_0^\infty e^{-t} t^{3/2} dt$

<u>Solution</u> since $\frac{3}{2} = \frac{3}{2} + 1 - 1 = 2.5 - 1$ then $\int_0^\infty e^{-t} t^{3/2} dt = \int_0^\infty e^{-t} t^{2.5-1} dt$

then $\int_0^\infty e^{-t}t^{2.5-1}dt = \Gamma(2.5)$ to find $\Gamma(2.5)$ we use equation (*) $\Gamma(\propto) = \frac{\Gamma(\propto+1)}{\propto}$

$\propto\Gamma(\propto) = \Gamma(\propto+1)$ let $\propto = 1.5$ then $= \Gamma(2.5) = 1.5\,\Gamma(1.5) = 1.5 \times 0.886227$

so $\Gamma(2.5) = 1.32934$

Example:- What is $\Gamma(-2.7)$

Solution From general form of equation (*) which is $\Gamma(\propto) = \frac{\Gamma(\propto+k+1)}{\propto(\propto+1)(\propto+2)\cdots\cdots(\propto+k)}$

let $\propto = -2.7$ and $k = 3$ then $\Gamma(-2.7) = \frac{\Gamma(-2.7+3+1)}{-2.7(-2.7+1)(-2.7+2)(-2.7+3)}$

$$= \frac{\Gamma(1.3)}{-0.9639} = \frac{0.897471}{-0.9639} = -0.93108$$

Example:- Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Solution from the definition of Gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t}\,t^{\propto-1}dt$.

if $\propto = \frac{1}{2}$ then $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}dt \Rightarrow \Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}\,t^{-\frac{1}{2}}dt$ ………… (1)

let $t = u^2$ then $dt = 2udu$ equation (1) becomes to

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-u^2}\,u^{-1}2udu \Rightarrow \Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du \quad \cdots\cdots\cdots\cdots (a)$$

and let $t = v^2$ then $dt = 2vdv \Rightarrow \Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-v^2}\,dv$ ………… (b)

now multiplying equations (a) and (b)

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4\int_0^\infty\int_0^\infty e^{-v^2}\,e^{-u^2}dudv = 4\int_0^\infty\int_0^\infty e^{-(v^2+u^2)}\,dudv$$

Using polar coordinates , where $v^2 + u^2 = r^2$ and $dudv = rdrd\theta$

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta = 4\frac{\pi}{2}\int_0^\infty e^{-r^2}rdr = -\pi \quad e^{-r^2}\Big]_0^\infty = \pi$$

$\therefore$ $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$