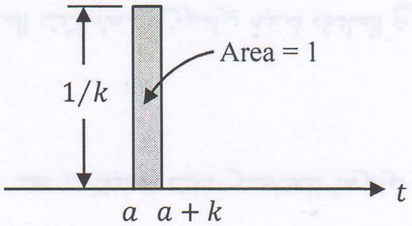
Dirac's Delta Function

Consider the function

$$f_k(t-a) = \begin{cases} 1/k & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases} \quad \cdots\cdots\cdots (1)$$

This function represents a force of magnitude $1/k$ acting from $t=a$ to $t=a+k$, where k is positive and small. The integral of a function acting over a time interval $a \le t \le a+k$ is called the **impulse** of the function.



Now, the impulse of f_k is

$$I_k = \int_0^{\infty} f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1 \quad \cdots\cdots\cdots (2)$$

By taking the limit of f_k as $k \to 0$

$$\lim_{k\to 0} f_k(t-a) = \delta(t-a)$$

$\delta(t-a)$ is called **Dirac delta function**

Note :- From equations (1) and (2) by taking limit as $k \to 0$ we obtain

$$\delta(t-a) = \begin{cases} \infty & if \;\; t=a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^{\infty} \delta(t-a)\,dt = 1$$

Sifting property of $\delta(t-a)$

$$\boxed{\int_0^{\infty} g(t)\,\delta(t-a)\,dt = g(a)}$$

To obtain the L.T of $\delta(t-a)$, we write

$$f_k(t-a) = \frac{1}{k}[u(t-a) - u(t-(a+k))]$$

and take L.T $\quad$ L.T of $\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\,\frac{1-e^{-ks}}{ks}$

now, taking the limit as $k \to 0$ (using l'Hopital's rule)

$$\lim_{k\to 0} \frac{se^{-(a+k)s}}{s} = e^{-as}$$

Then $\quad \boxed{\text{L.T of } \delta(t-a) = e^{-as}}$

Example:-

Find the response of the mass - spring – damper system under a unit impulse (Dirac delta function) at time $t = 1$. Let m = 1 kg , c = 3 and k = 2

Solution

The equation of motion in this case becomes $y''(t) + 3y'(t) + 2y(t) = \delta(t-1)$ Note that initial conditions = 0

Taking L.T of both sides of differential equation

$$(s^2 + 3s + 2)Y(s) = e^{-s} \Rightarrow \quad Y(s) = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{k_1}{s+1} + \frac{k_2}{s+2}\right)e^{-s}$$

$$Y(s) = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}$$

By corollary 2 the $y(t) = \text{L.T}^{-1}Y(s)$ is

$$y(t) = e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)$$

$$y(t) = \begin{cases} 0 & 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & if \quad t > 1 \end{cases}$$

Convolution. Integral Equations

If L.T of $\{f(t)\} = F(s)$ and L.T of $\{g(t)\} = G(s)$ then

L.T of $\{fg\} \neq F(s) \cdot G(s)$

or the transform of a product is generally different from the product of the transforms of the factors. to see this consider $f(t) = e^t$ and $g(t) = 1$. Then $fg = e^t$, L.T of $\{fg\} = \frac{1}{s-1}$

but L.T of $\{f(t)\} = \frac{1}{s-1}$ and L.T of $\{g(t)\} = \frac{1}{s}$ so $F(s) \cdot G(s) = \frac{1}{s^2-s}$

$\therefore$ L.T of $\{fg\} \neq F(s) \cdot G(s)$

then, what is $F(s) \cdot G(s)$? The answer is

$F(s) \cdot G(s)$ is the transform of the **convolution** of $f(t)$ and $g(t)$, denoted by the standard notation $f * g$ which defined by the following integral

$$\boxed{h(t) = (f * g)t = \int_0^t f(\tau)g(t-\tau)\, d\tau}$$

Example:-

If $H(s) = 1/[(s-a)s]$. Find $h(t)$.

Solution

$1/(s-a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$

Now $f(\tau) = e^{a\tau}$ and $g(t-\tau) = 1$

Then $h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1\, d\tau = \frac{1}{a}(e^{at} - 1)$

<u>Properties of Convolution</u>

$$f * g = g * f$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

$$(f * g) * v = f * (g * v)$$

$$f * 0 = 0 * f = 0$$

Unusual Properties of Convolution

$$f * 1 \neq f$$

Let $f = t$ then $t * 1 = \int_0^t \tau \cdot 1\, d\tau = \frac{1}{2}t^2 \neq t$

<u>Example</u>:- Find the solution of the differential equation $y''(t) + 4y'(t) + 13y(t) = \frac{1}{3}e^{-2t} \sin 3t$ for which $y(0) = 1,$ $y'(0) = -2.$

<u>Solution</u>

Taking L.T of both sides of equation

$$s^2Y(s) - sy(0) - y'(0) + 4[s\,Y(s) - y(0)] + 13Y(s) = \frac{1}{3}\text{L.T } of\ (e^{-2t} \sin 3t)$$

$$s^2Y(s) - s + 2 + 4[s\,Y(s) - 1] + 13Y(s) = \frac{1}{3}\ \frac{3}{(s+2)^2+3^2}$$

$$(s^2 + 4s + 13)Y(s) = \frac{1}{(s+2)^2+3^2} + s + 2 \quad \Rightarrow \quad [(s+2)^2 + 3^2]Y(s) = \frac{1}{(s+2)^2+3^2} + s + 2$$

$$Y(s) = \frac{1}{((s+2)^2+3^2)^2} + \frac{s+2}{(s+2)^2+3^2}$$

$$y(t) = \text{LT}^{-1}\frac{1}{((s+2)^2+3^2)^2} + \text{LT}^{-1}\frac{s+2}{(s+2)^2+3^2}$$

$$y(t) = e^{-2t}\text{LT}^{-1}\frac{1}{(s^2+3^2)^2} + e^{-2t}\cos 3t$$

now, to find the $\text{LT}^{-1}\frac{1}{(s^2+3^2)^2}$ by convolution , where $\frac{1}{(s^2+3^2)^2} = \text{L.T of}\left\{\frac{\sin 3t}{3}\right\}\text{L.T of}\left\{\frac{\sin 3t}{3}\right\}$

then $\text{LT}^{-1}\frac{1}{(s^2+3^2)^2} = \frac{1}{9}\int_0^t \sin 3(t-\tau) \cdot \sin 3\tau \cdot d\tau$

but $\sin x \sin y = \frac{1}{2}[-\cos(x+y) + \cos(x-y)]$

$$\sin 3(t-\tau) \cdot \sin 3\tau = \frac{1}{2}[-\cos 3t + \cos(3t - 3\tau - 3\tau)] = \frac{1}{2}[\cos(3t - 6\tau) - \cos 3t]$$

$$= \frac{1}{2}[\cos(6\tau - 3t) - \cos 3t]$$

so $\quad LT^{-1}\frac{1}{(s^2+3^2)^2} = \frac{1}{18}\int_0^t [\cos(6\tau - 3t) - \cos 3t]\, d\tau = \frac{1}{18}\left\{\frac{\sin(6\tau-3t)}{6}\bigg|_0^t - \tau\cos 3t\big|_0^t\right\}$

$$= \frac{1}{18}\left\{\frac{1}{6}[\sin 3t + \sin 3t] - t\cos 3t\right\} = \frac{\sin 3t - 3t\cos 3t}{54}$$

$$y(t) = e^{-2t}\ \frac{\sin 3t - 3t\cos 3t}{54} + e^{-2t}\ \cos 3t$$

Example:- What is y(t) if $Y(s) = \frac{s+2}{(s^2+4s+5)^2}$

Solution

$$s^2 + 4s + 5 + 4 - 4 = s^2 + 4s + 4 + 1 = (s+2)^2 + 1$$

so $Y(s)$ can be written as $\quad Y(s) = \frac{s+2}{[(s+2)^2+1]^2}$

$$LT^{-1} \text{ of } \{Y(s)\} = e^{-2t} LT^{-1}\frac{s}{[s^2+1]^2}$$

Method I

$$\frac{s}{[s^2+1]^2} = [s^2+1]^{-2}\ s$$

then from integration theorem $\quad \left\{\frac{f(t)}{t}\right\} = LT^{-1} \text{ of } \int_s^\infty \phi(s)\, ds$

let $\quad \phi(s) = [s^2+1]^{-2}\ s\ \cdot \frac{2}{2} \quad , \quad \int_s^\infty \phi(s)\, ds = \frac{1}{2}\int_s^\infty [s^2+1]^{-2}\ 2s\, ds = -\frac{1}{2}\frac{1}{s^2+1}\bigg|_s^\infty$

$\int_s^\infty \phi(s)\, ds = \frac{1}{2}\frac{1}{s^2+1} \qquad f(t) = \frac{1}{2}\ t\ \sin t$

then $\quad y(t) = \frac{1}{2}\, t\, e^{-2t}\sin t$

Method II (by convolution theorem)

since $\quad \frac{s}{[s^2+1]^2} = \underbrace{\frac{1}{s^2+1}}_{F(s)} \cdot \underbrace{\frac{s}{s^2+1}}_{G(s)} \qquad$ where $f(t) = \sin t \qquad g(t) = \cos t$

now $\quad f * g = \int_0^t \sin\tau \cdot \cos(t-\tau)\, d\tau$

but $\quad \sin x\cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

let $\quad x = \tau \ , \quad y = t - \tau \quad$ then $\quad x + y = t \quad$ and $\quad x - y = 2\tau - t$

$$f * g = \int_0^t \sin\tau \cdot \cos(t-\tau)\, d\tau = \frac{1}{2}\int_0^t [\sin t + \sin(2\tau - t)]\ d\tau$$

$$= \frac{1}{2}[\tau \sin t|_0^t] + \frac{1}{2}[-\cos(2\tau - t)|_0^t] = \frac{1}{2} t\sin t$$

$$y(t) = \frac{1}{2}\ t\ e^{-2t}\sin t$$

Example:- Solve the integral equation

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)d\tau = t$$

Solution

Write the given equation as a convolution $y(t) - y(t) * \sin t = t$

taking L.T of both sides $Y(s) - Y(s)\frac{1}{s^2+1} = \frac{1}{s^2}$ $\Rightarrow$ $Y(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s^2}$

$$Y(s)\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s^2} \quad \Rightarrow \quad Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

then $y(t) = t + \frac{t^3}{3!}$

Example:- Solve the integral equation

$$y(t) - \int_0^t (1+\tau)\, y(t-\tau)\, d\tau = 1 - \sinh t$$

Solution

Write the given equation as a convolution $y(t) - (1+t) * y(t) = 1 - \sinh t$

taking L.T of both sides $Y(s) - \left(\frac{1}{s} + \frac{1}{s^2}\right)Y(s) = \frac{1}{s} - \frac{1}{s^2-1}$ $\Rightarrow$ $Y(s)\left[\frac{s^2-s-1}{s^2}\right] = \frac{s^2-s-1}{s(s^2-1)}$

$$\Rightarrow \quad Y(s) = \frac{s}{s^2-1}$$

then $y(t) = \cosh t$

Problems

Convolution By Integration

1- $t * e^t$

2- $e^{at} * e^{bt}$ $(a \neq b)$

3- $1 * \cos \omega t$

Using Convolution Theorem, Solve

4- $y''(t) + 5y'(t) + 4y(t) = 2e^{-2t}$ for which $y(0) = 0$ $y'(0) = 0$

5- $y''(t) + 4y(t) = 5u(t-1)$ for which $y(0) = 0$ $y'(0) = 0$

6- $y''(t) + 5y'(t) + 6y(t) = \delta(t-3)$ for which $y(0) = 1$ $y'(0) = 0$

7- $y''(t) + 6y'(t) + 8y(t) = 2\delta(t-1) + 2\delta(t-2)$ for which $y(0) = 1$ $y'(0) = 0$

Integral Equation

8- $y(t) - \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = t + e^t$

9- $y(t) - \int_0^t y(\tau)\,\sin(t-\tau)\,d\tau = \cos t$

Problems

Find the L.T of each of the following functions:

1- $u(t-a)$

2- $u(t-e^{-t})$

3- $t^2u(t-2)$

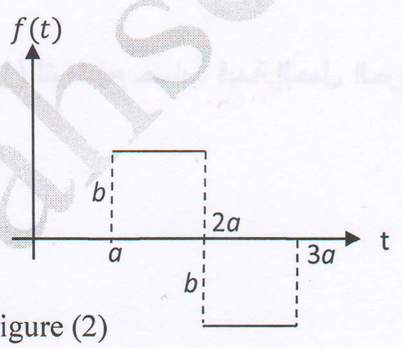
4- $\cos t\,u(t-1)$

5- $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t \end{cases}$

6- $(t) = \begin{cases} t & 0 < t < 2 \\ 2 & 2 < t \end{cases}$

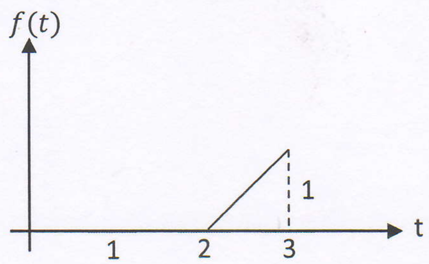
7- The function graphed in Figure (1)

Figure (1)



8- The function graphed in Figure (2)

Figure (2)



9- $te^{-3t}\,\sin 2t$

10- $e^{-3t}\int_0^t t\,\sin 2t\,\,dt$

11- $t\int_0^t e^{-3t} \sin 2t \ \ dt$

12- $\int_0^t te^{-3t} \sin 2t \ \ dt$

13- $\frac{1-\cos 3t}{t}$

14- $\frac{e^{-3t} \sin 2t}{t}$

15- $e^{-3t}\int_0^t \frac{\sin 2t}{t} \ dt$

Find the inverse of each of the following transform

16- $\frac{1}{(s+2)^4}$

17- $\frac{s+1}{9s^2+6s+5}$

18- $\frac{1}{s^2(s+1)}$

19- $\frac{1}{s^4+5s^2+4}$

20- $\frac{e^{-3s}}{s^2-9}$

21- $\frac{e^{-s}+e^{-2s}}{s^2-3s+2}$

22- $\ln\frac{s^2-1}{s^2}$

23- $\frac{2}{(s^2+4)^2}$

24- $\frac{s+2}{(s^2+4s+5)^2}$

25- $\frac{1}{s}\tan^{-1}\frac{1}{s}$

26- $\frac{1}{(s^2+2s+2)^2}$

Use the Laplace Transformation to solve the variable-coefficient linear differential equation

27- $ty''(t) + 2(t-1)y'(t) + (t-2)y(t) = 0$ Answer :- $y = y_o\, e^{-t} + \frac{c}{6}\, t^3 e^{-t}$

28- $ty''(t) + 2(2t-1)y'(t) + 4(t-1)y(t) = 0$

29- $ty''(t) + 2y'(t) + ty(t) = 0$

30- $ty''(t) + 2(2t-1)y'(t) + 4y(t) = 0$