Laplace Transform **(Simon Laplace 1749 – 1827 was a great French mathematician)**

Is the transformation the independent variable to s domain, if the independent variable is t then

$$t_{domain} \rightarrow s_{domain}$$

Definition:- Laplace transform L.T of $f(t)$ is

$$F(s) = \int_0^\infty f(t) \quad e^{-st} \quad dt$$

where s is complex variable or $s = x + iy$, $i = \sqrt{-1}$

Example:- what is the L.T of $f(t) = 1$

Solution From the definition L.T of $f(t) = \int_0^\infty e^{-st} \ f(t) \ dt$

$$= \int_0^\infty e^{-st} \ dt = -\frac{1}{s} \ e^{-st}|_0^\infty = -\frac{1}{s}(e^{-\infty} - e^0) = \boxed{\frac{1}{s}}$$

Example:- What is the L.T of $f(t) = t$

Solution L.T of $f(t) = \int_0^\infty t \, e^{-st} \ dt$ $\Rightarrow$ Let u = t $\Rightarrow$ du = dt

dv = $e^{-st}dt$ $\Rightarrow$ v = $-\frac{1}{s} e^{-st}$

so $$\int_0^\infty t \, e^{-st} \ dt = \underbrace{-t \frac{1}{s} \ e^{-st}|_0^\infty}_{zero} + \int_0^\infty \frac{1}{s} e^{-st} \ dt = -\frac{1}{s^2} \ e^{-st}|_0^\infty = \boxed{\frac{1}{s^2}}$$

Example:- Find the L.T of $f(t) = e^{at}$

Solution L.T of $f(t) = \int_0^\infty e^{at} \, e^{-st} \ dt = \int_0^\infty e^{(a-s)t} \ dt = \frac{1}{a-s} \ e^{(a-s)t}|_0^\infty$

$$= \frac{1}{a-s}(e^{-\infty} - e^0) = \boxed{\frac{1}{s-a}}$$ where $s > a$

The General Method

The utility of the Laplace transform is based primarily upon the following three theorems

Theorem 1:-

$$\text{L.T of } [c_1 f_1(t) \pm c_2 f_2(t)] = c_1 F_1(s) \pm c_2 F_2(s)$$

Prove

$$\text{L.T of } [c_1 f_1(t) \pm c_2 f_2(t)] = \int_0^\infty [c_1 f_1(t) \pm c_2 f_2(t)] e^{-st} \ dt$$

$$= \int_0^\infty c_1 f_1(t) e^{-st} \ dt \pm \int_0^\infty c_2 f_2(t) e^{-st} \ dt = c_1 \int_0^\infty f_1(t) e^{-st} \ dt \pm c_2 \int_0^\infty f_2(t) e^{-st} \ dt$$

$$= c_1 F_1(s) \pm c_2 F_2(s)$$

Example:- Find the L.T of $f(t) = \cosh at$

Solution Since $\cosh at = \frac{1}{2}[e^{at} + e^{-at}]$

$$\text{L.T of } \cosh at = \frac{1}{2}\int_0^\infty e^{at} \, e^{-st} \ dt + \frac{1}{2}\int_0^\infty e^{-at} \, e^{-st} \ dt$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$$

$\therefore$ L.T of $\cosh at = \boxed{\frac{s}{s^2-a^2}}$

H.W Prove that L.T of $\sinh at = \frac{a}{s^2-a^2}$

Example:- Find the L.T of $f(t) = \cos\omega t$

Solution From Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

then replace θ by $\omega t \Rightarrow e^{i\omega t} = \cos\omega t + i\sin\omega t$ ………… (1)

and replace θ by $-\omega t \Rightarrow e^{-i\omega t} = \cos\omega t - i\sin\omega t$ ………… (2)

adding equations (1) and (2) $\cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$

now, L.T of $\cos\omega t = \frac{1}{2}\int_0^\infty (e^{i\omega t} + e^{-i\omega t})e^{-st}\,dt = \frac{1}{2}\int_0^\infty e^{(i\omega-s)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(i\omega+s)t}\,dt$

$= \frac{1}{2}\frac{1}{i\omega-s}e^{(i\omega-s)t}\Big|_0^\infty - \frac{1}{2}\frac{1}{i\omega+s}e^{-(i\omega+s)t}\Big|_0^\infty$

If $s > \omega$ then L.T of $\cos\omega t = -\frac{1}{2}\,\frac{1}{i\omega-s} + \frac{1}{2}\,\frac{1}{i\omega+s} = \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$

$= \frac{1}{2}\left(\frac{s+i\omega+s-i\omega}{s^2+\omega^2}\right) = \boxed{\frac{s}{s^2+\omega^2}}$

H.W Prove that L.T of $\sin\omega t = \boxed{\frac{\omega}{s^2+\omega^2}}$

Example:- Find the L.T of $f(t) = t^n$, $n > -1$

Solution L.T of $t^n = \int_0^\infty t^n e^{-st}\,dt$, let $st = z$ $t = \frac{z}{s} \Rightarrow dt = \frac{dz}{s}$

L.T of $t^n = \int_0^\infty \left(\frac{z}{s}\right)^n e^{-z}\frac{dz}{s} = \frac{1}{s^{n+1}}\int_0^\infty z^n e^{-z}dz = \frac{1}{s^{n+1}}\int_0^\infty z^{n+1-1}e^{-z}dz$

Then L.T of $t^n = \boxed{\frac{\Gamma(n+1)}{s^{n+1}}}$

Now, if n is positive integer $\Rightarrow$ $\Gamma(n+1) = n!$ L.T of $t^n = \boxed{\frac{n!}{s^{n+1}}}$

Example:- If $F(s) = \frac{s+1}{s^2+s-6}$ what is $y(t)$?

Solution

$$\frac{s+1}{s^2+s-6} = \frac{s+1}{(s-2)(s+3)} = \frac{k_1}{s-2} + \frac{k_2}{s+3}$$

Where $k_1 = \frac{s+1}{s+3}\Big|_{s=2} = \frac{3}{5}$ $k_2 = \frac{s+1}{s-2}\Big|_{s=-3} = \frac{2}{5}$

$F(s) = \frac{1}{5}\left(\frac{3}{s-2} + \frac{2}{s+3}\right)$

$y(t) = \frac{1}{5}[3e^{2t} + 2e^{-3t}]$

Theorem 2:-

$$\boxed{\text{L.T of } \{f'(t)\} = \text{L.T of } \left\{\frac{df}{dt}\right\} = s\,F(s) - f(0)}$$

Example:- What is L.T of $\{f''(t)\}$

Solution Since $f''(t) = [f'(t)]'$ let $f'(t) = g(t)$

Then $f''(t) = [g(t)]'$ so L.T of $\{f''(t)\} =$ L.T of $[g(t)]'$

From Theorem 2 L.T of $[g(t)]' = sG(s) - g(0) = s$ L.T of $\{f'(t)\} - f'(0)$

Applying Theorem 2 again L.T of $\{f''(t)\} = s[sF(s) - f(0)] - f'(0)$

$\therefore$ L.T of $\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

H.W Prove that L.T of $\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

Example:- Find the particular solution of the differential equation $y''(t) - 3y'(t) + 2y(t) = 12e^{-2t}$ for which $y(0) = 2, \quad y'(0) = 6$

Solution From theorem 2 L.T of $\{f'(t)\} = s\,F(s) - f(0)$

L.T of $\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

Taking L.T of both side of equation

$$[s^2\,Y(s) - sy(0) - y'(0)] - 3\,[sY(s) - y(0)] + 2Y(s) = 12\,\text{L.T}[e^{-2t}]$$

Substitute $y(0) = 2$ and $y'(0) = 6$ $\qquad \text{L.T}[e^{-2t}] = \frac{1}{s+2}$

$$[s^2\,Y(s) - 2s - 6] - 3\,[sY(s) - 2] + 2Y(s) = \frac{12}{s+2}$$

$$s^2\,Y(s) - 2s - 6 - 3sY(s) + 6 + 2Y(s) = \frac{12}{s+2}$$

$$(s^2 - 3s + 2)Y(s) = 2s + \frac{12}{s+2} = \frac{2s^2+4s+12}{s+2}$$

$$Y(s) = \frac{2s^2+4s+12}{(s^2-3s+2)(s+2)} = \frac{2s^2+4s+12}{(s-1)(s-2)(s+2)} = \frac{k_1}{s-1} + \frac{k_2}{s-2} + \frac{k_3}{s+2}$$

Where $k_1 = -6 \quad , \quad k_2 = 7 \quad , \quad k_3 = 1$

$$Y(s) = \frac{-6}{s-1} + \frac{7}{s-2} + \frac{1}{s+2}$$

$$y(t) = -6\,e^t + 7\,e^{2t} + e^{-2t}$$

Theorem 3:-

$$\text{L.T of } \left\{\int_a^t f(t)\,dt\right\} = \frac{1}{s}\,F(s) + \frac{1}{s}\int_a^0 f(t)\,dt \quad , \quad a \geq 0$$

Example:- Show that L.T of $\left[\int_a^t \int_a^t f(t)dtdt\right] = \frac{1}{s^2}\,F(s) + \frac{1}{s^2}\int_a^0 f(t)\,dt + \frac{1}{s}\int_a^0 \int_a^t f(t)dtdt$

Solution

Let $\int_a^t f(t)\,dt = g(t)$

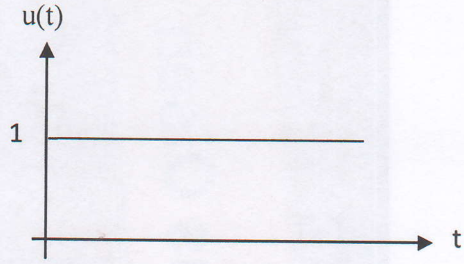
then L.T of $\left[\int_a^t \int_a^t f(t)dtdt\right] =$ L.T of $\int_a^t g(t)\,dt = \frac{1}{s}\,G(s) + \frac{1}{s}\int_a^0 g(t)\,dt$

$$= \frac{1}{s}\left[\text{L.T of } \int_a^0 f(t)\,dt\right] + \frac{1}{s}\int_a^0 \int_a^t f(t)dtdt = \frac{1}{s}\left[\frac{1}{s}F(s) + \frac{1}{s}\int_a^0 f(t)\,dt\right] + \frac{1}{s}\int_a^0 \int_a^t f(t)dtdt$$

$$\therefore \quad \text{L.T of } \left[\int_a^t \int_a^t f(t)dtdt\right] = \frac{1}{s^2}\,F(s) + \frac{1}{s^2}\int_a^0 f(t)\,dt + \frac{1}{s}\int_a^0 \int_a^t f(t)dtdt$$

Unit Step Function

The unit step function can be defined as



$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Example:- Solve for $y(t)$ from the simultaneous equations

$$y'(t) + 2y(t) + 6\int_0^t z\,dt = -2\ u(t) \quad \cdots\cdots\cdots\ (1)$$

$$y'(t) + z'(t) + z = 0 \quad \cdots\cdots\cdots\ (2)$$

where $y(0) = -5 \qquad z(0) = 6$

Solution

Taking L.T of each equation term by term

Equation 1

$$[s\,Y(s) - (-5)] + 2Y(s) + 6\frac{1}{s}Z(s) = -2\cdot\frac{1}{s}$$

Note :- since $a = 0$, then $\int_a^0 z(t)\,dt = 0$

$$s\,Y(s) + 2Y(s) + \frac{6}{s}Z(s) = -\frac{2}{s} - 5 \quad \Rightarrow \quad (s^2+2s)Y(s) + 6Z(s) = -2 - 5s \quad \cdots\cdots\cdots\ (3)$$

Equation 2

$$s\,Y(s) + 5 + sZ(s) - 6 + Z(s) = 0 \quad \Rightarrow \quad s\,Y(s) + (s+1)Z(s) = 1 \quad \cdots\cdots\cdots\ (4)$$

Equations (3) and (4) can be written as a matrix form

$$\begin{bmatrix} s^2+2s & 6 \\ s & s+1 \end{bmatrix}\begin{bmatrix} Y(s) \\ Z(s) \end{bmatrix} = \begin{bmatrix} -2-5s \\ 1 \end{bmatrix}$$

Applying the Grammar rule's

$$Y(s) = \frac{\begin{vmatrix} -2-5s & 6 \\ 1 & s+1 \end{vmatrix}}{\begin{vmatrix} s^2+2s & 6 \\ s & s+1 \end{vmatrix}} = \frac{(s+1)(-2-5s)-6}{(s^2+2s)(s+1)-6s} = \frac{-5s^2-7s-8}{s^3+3s^2-4s}$$

$$Y(s) = \frac{-5s^2-7s-8}{s(s-1)(s+4)} = \frac{k_1}{s} + \frac{k_2}{s-1} + \frac{k_3}{s+4}$$

Where $k_1 = 2$, $\quad k_2 = -4 \quad$, $\quad k_3 = -3$

$$Y(s) = \frac{2}{s} - \frac{4}{s-1} - \frac{3}{s+4} \quad \Rightarrow \quad y(t) = 2u(t) - 4\,e^{t} - 3e^{-4t}$$

Example:- What is L.T of $f(t) = \sin^2 t$

Solution1

$$\sin^2 t = \frac{1}{2} - \frac{\cos 2t}{2}$$

$$\therefore \quad \text{L.T of } \sin^2 t = \frac{1}{2s} - \frac{s}{2(s^2+4)} = \frac{s^2+4-s^2}{2s(s^2+4)} = \frac{2}{s(s^2+4)}$$

Solution2

$$f(t) = \sin^2 t \qquad \Rightarrow \qquad f'(t) = 2\sin t\cos t = \sin 2t$$

since $\quad$ L.T of $\{f'(t)\} = s\,F(s) - f(0)$

L.T of $\{\sin 2t\} = s[\text{L.T of } f(t)] - 0$

$$\Rightarrow \quad \frac{2}{s^2+2^2} = s[\text{L.T of } f(t)] \qquad \therefore \qquad [\text{L.T of } f(t)] = \text{L.T of } \sin^2 t\,] = \frac{2}{s(s^2+4)}$$

H.W What is L.T of the following

1- $\cos(at+b)$ $\quad$ 2- $\cos^2(bt)$ $\quad$ 3- $(t+1)^2$

Theorem :- If a Laplace Transform contains the factor s, the inverse of that transform can be found by suppressing the factor s, determining the inverse of the remaining portion of the transform, and finally differentiating that inverse with respect to t.

$$f(t) = \frac{d}{dt}\ \text{L.T}^{-1}\{\phi(s)\}$$

Example:- What is $\text{L.T}^{-1}\left[\frac{s}{s^2+4}\right]$

Solution $\quad \frac{s}{s^2+4} = \frac{s}{s^2+2^2}$ suppressing the factor s $\Rightarrow \quad \phi(s) = \frac{1}{s^2+2^2} = \frac{1}{2}\frac{2}{s^2+2^2}$

$$f(t) = \frac{d}{dt}\ \text{L.T}^{-1}\left\{\frac{1}{2}\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\frac{d}{dt}\text{L.T}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = = \frac{1}{2}\frac{d}{dt}\sin 2t = \cos 2t$$

Theorem :- If a Laplace Transform contains the factor $\frac{1}{s}$, the inverse of that transform can be found by suppressing the factor $\frac{1}{s}$,, determining the inverse of the remaining portion of the transform, and finally integrating that inverse with respect to t from $0 \to t$.

$$f(t) = \int_0^t \text{L.T}^{-1}\{\phi(s)\}\,dt$$

Example:- What is $\text{L.T}^{-1}\left[\frac{1}{s^3+4s}\right]$

Solution $\quad \text{L.T}^{-1}\left[\frac{1}{s^3+4s}\right] = \text{L.T}^{-1}\left[\frac{1}{s(s^2+4)}\right] \quad \Rightarrow \quad \phi(s) = \frac{1}{s^2+2^2}$

$$f(t) = \int_0^t \text{L.T}^{-1}\{\phi(s)\}\,dt = f(t) = \int_0^t \text{L.T}^{-1}\frac{1}{2}\ \frac{2}{s^2+2^2}dt$$

$$f(t) = \frac{1}{2}\ \int_0^t \sin 2t\,dt = -\frac{1}{4}\ \cos 2t]_0^t = \frac{1}{4} - \frac{\cos 2t}{4} = \frac{1-\cos 2t}{4} = \frac{1}{2}\ \sin^2 t$$

First Shifting Theorem:- This theorem says that the Transform of e^{-at} times a function of t is equal to the transform of the function itself , with s replaced by $s+a$

$$\text{LT}\{e^{-at}f(t)\} = \text{LT}\{f(t)\}|_{s\to s+a}$$

By means of this theorem we can easily establish the following important formulas:-

Formula 1 :- $\quad \text{LT}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$

Formula 2 :- $\mathrm{LT}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$

Formula 3 :- $\mathrm{LT}\{e^{-at}t^n\} = \begin{cases} \frac{\Gamma(n+1)}{(s+a)^{n+1}} & n > -1 \\ \frac{n!}{(s+a)^{n+1}} & n\text{: a positive integer} \end{cases}$

Corollary $[\mathrm{L}\cdot\mathrm{T}]^{-1}\{\phi(s)\} = e^{-at}[\mathrm{L}\cdot\mathrm{T}]^{-1}\{\phi(s-a)\}$

This theorem says that, if we replace $s+a$ by s or s by $s-a$ in the transform of a function, then the inverse of the modified transform $\phi(s-a)$ must be multiplied by e^{-at} to obtain the inverse of the original transform $\phi(s)$.

Example:- If L.T of $y(t)$ is $Y(s) = \frac{2s+5}{s^2+4s+13}$ what is $y(t)$

Solution

$$\frac{2s+5}{s^2+4s+13} = \frac{(2s+4)+1}{s^2+4s+13+4-4} = \frac{2(s+2)+1}{s^2+4s+4+9} = \frac{2(s+2)+1}{(s+2)^2+3^2}$$

$$Y(s) = \frac{2s+5}{s^2+4s+13} = \frac{2(s+2)}{(s+2)^2+3^2} + \frac{1}{(s+2)^2+3^2}\,\frac{3}{3}$$

$$y(t) = \mathrm{LT}^{-1}Y(s) = 2e^{-2t}\cos 3t + \frac{1}{3}e^{-2t}\sin 3t$$

Example:- What is the solution of the differential equation $y''(t) + 2y'(t) + y(t) = te^{-t}$ for which $y(0) = 1,\quad y'(0) = -2$

Solution Taking L.T of both sides of differential equation

$$s^2Y(s) - sy(0) - y'(0) + 2[s\,Y(s) - y(0)] + Y(s) = \mathrm{L.T}\ of\ (te^{-t}) \cdots\cdots\cdots (1)$$

but $\mathrm{L.T}\ of\ (te^{-t}) = \frac{1}{(s+1)^2}$ equation (1) becomes to

$$s^2Y(s) - s + 2 + 2[s\,Y(s) - 1] + Y(s) = \frac{1}{(s+1)^2}$$

$$(s^2 + 2s + 1)Y(s) = \frac{1}{(s+1)^2} + s \quad\Rightarrow\quad (s+1)^2Y(s) = \frac{1}{(s+1)^2} + s$$

$Y(s) = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}$ then $y(t) = \mathrm{L.T}^{-1}Y(s)$

$$\mathrm{L.T}^{-1}\left\{\frac{1}{(s+1)^4}\right\} = \frac{1}{3!}e^{-t}t^3$$

and $\mathrm{L.T}^{-1}\left\{\frac{s}{(s+1)^2}\right\}$

Method 1

$$\frac{s}{(s+1)^2} = \frac{s+1-1}{(s+1)^2} = \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

then $\mathrm{L.T}^{-1}\left\{\frac{1}{s+1} - \frac{1}{(s+1)^2}\right\} = e^{-t} - te^{-t}$

Method 2

Suppressing s from $\frac{s}{(s+1)^2}$ then , $\phi(s) = \frac{1}{(s+1)^2}$