then $\quad L.T^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t} \quad , \qquad \frac{d}{dt} \; L.T^{-1}\{\phi(s)\} = e^{-t} - te^{-t}$

$\therefore \qquad y(t) = \frac{1}{3!}e^{-t}t^3 + e^{-t} - te^{-t}$

Theorem :- Initial value theorem

Using this theorem we can find the initial value of a of a function without finding the complete solution

$$f(0^+) = \lim_{s\to\infty}[sF(s)]$$

Prove

$$L.T\ of\ \{f'(t)\} = \int_0^\infty f'(t)\ e^{-st}\ dt = s\,F(s) - f(0)$$

Taking limit as $s \to \infty$ of both sides of above equation

$$\lim_{s\to\infty}\int_0^\infty f'(t)\ e^{-st}\ dt = \lim_{s\to\infty}[s\,F(s) - f(0)]$$

as $\quad s \to \infty \qquad e^{-st} \to 0 \qquad \Rightarrow \qquad \lim_{s\to\infty}[s\,F(s) - f(0)] = 0$

since $f(0)$ is constant $\qquad f(0) = \lim_{s\to\infty}[sF(s)]$

<u>Example</u>:- If $\quad Y(s) = \frac{-5s^2-7s-8}{s^3+3s^2-4s} \qquad$ what is $y(0)$

<u>Solution</u>

$$sF(s) = \frac{-5s^3-7s^2-8s}{s^3+3s^2-4s}$$

$$\lim_{s\to\infty}[sF(s)] = \frac{-5-\frac{7}{s}-\frac{8}{s^2}}{1+\frac{3}{s}-\frac{4}{s^2}} = -5$$

The student can check this result by taking $[L\cdot T]^{-1}$ then taking $\lim_{t\to 0} y(t)$

Corollary

$$\lim_{s\to\infty} s[sF(s) - f(0^+)] = f'(0^+)$$

<u>Example</u>:- If

$$F(s) = \frac{s+3}{2s^2+2s+1}$$

what are the values of $f(0^+)$ and $f'(0^+)$?

<u>Solution</u>

$$sF(s) = \frac{s^2+3s}{2s^2+2s+1}$$

$$f(0^+) = \lim_{s\to\infty}[sF(s)] = \lim_{s\to\infty}\left[\frac{s^2+3s}{2s^2+2s+1}\right] = \lim_{s\to\infty}\left[\frac{1+\frac{3}{s}}{2+\frac{2}{s}+\frac{1}{s^2}}\right] = \frac{1}{2}$$

$$f'(0^+) = \lim_{s\to\infty} s[sF(s) - f(0^+)] = \lim_{s\to\infty} s\left[\frac{s^2+3s}{2s^2+2s+1} - \frac{1}{2}\right]$$

$$f'(0^+) = \lim_{s\to\infty} s\left[\frac{4s^2 - s}{2(2s^2 + 2s + 1)}\right] = 1$$

Theorem :- Final value theorem

$$f(\infty) = \lim_{s\to 0}[sF(s)]$$

Prove

$$\text{L.T } of\ \{f'(t)\} = \int_0^\infty f'(t)\ e^{-st}\ dt = s\,F(s) - f(0)$$

Taking limit as $s \to 0$ of both sides of above equation

$$\lim_{s\to 0}\int_0^\infty f'(t)\ e^{-st}\ dt = \lim_{s\to 0}[s\,F(s) - f(0)]$$

as $s \to 0$ $\quad e^{-st} = 1 \quad \Rightarrow \quad f(t)|_0^\infty = \lim_{s\to 0}[s\,F(s) - f(0)]$

$$\Rightarrow \quad f(\infty) - f(0) = \lim_{s\to 0}[s\,F(s)] - f(0)$$

$$f(\infty) = \lim_{s\to 0}[s\,F(s)]$$

<u>**Example**</u>:- If $\mathrm{Y}(s) = \frac{1}{s+1}$ what is $y(\infty)$

<u>Solution</u>

$$f(\infty) = \lim_{s\to 0}[s\,F(s)] = \lim_{s\to 0}\frac{s}{s+1} = 0$$

Check since $\mathrm{LT}^{-1}\frac{1}{s+1} = e^{-t}$ $\quad f(\infty) = \lim_{t\to\infty}[e^{-t}] = 0$

<u>**Differentiation and Integration Theorems of Transform**</u>

1-Differentiation Theorem

If $f(t)$ is piecewise regular on $[0,\infty]$ and of exponential order and if L.T of $f(t) = \phi(s)$ then:-

$$\text{L.T of } \{tf(t)\} = -\phi'(s)$$

Prove By definition we have

$$\text{L.T } of\ \{f(t)\} = F(s) = \int_0^\infty f(t)\ e^{-st}\ dt$$

Differentiating both sides with respect to s

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)\ e^{-st}\ dt$$

$$\frac{d}{ds}\phi(s) = \int_0^\infty \frac{\partial}{\partial s}f(t)\ e^{-st}\ dt = \int_0^\infty f(t)\,[-t\,e^{-st}]\ dt$$

Or $\quad \phi'(s) = -\int_0^\infty tf(t)\ e^{-st}\ dt = -\text{L.T of } \{tf(t)\}$

$\therefore \quad \text{L.T of } \{tf(t)\} = -\phi'(s)$

Corollary By taking inverses of above theorem and solve for $f(t)$ we obtain

$$\{tf(t)\} = -[\mathrm{L\cdot T}]^{-1}\{\phi'(s)\}$$

$$f(t) = -\frac{1}{t}[\mathrm{L\cdot T}]^{-1}\{\phi'(s)\}$$

<u>**Example**</u>:- Find is L.T of $\{t\ \sin\omega t\}$

<u>Solution</u>

$f(t) = \sin \omega t$ L.T of $\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2} = \phi(s)$

$\Rightarrow \quad \phi'(s) = \frac{-2s\omega}{(s^2+\omega^2)^2} \quad \Rightarrow \quad -\phi'(s) = \frac{2s\omega}{(s^2+\omega^2)^2}$

L.T of $\{t \sin \omega t\} = \frac{2s\omega}{(s^2+\omega^2)^2}$

HW:- Find is L.T of $\{t \cos \omega t\}$

Example:- Find is L.T of $\{t^2 \sin 4t\}$

Solution

$t^2 \sin 4t = t \cdot t \sin 4t = t\, f(t)$

But from previous example L.T of $\{t \sin 4t\} = \frac{8s}{(s^2+4^2)^2} = \phi(s)$

Then $\quad \phi'(s) = \frac{8(s^2+4^2)^2 - 8s^2 \cdot 4(s^2+4^2)}{(s^2+4^2)^4} = \frac{128-24s^2}{(s^2+4^2)^3}$

$\therefore \quad$ L.T of $\{t^2 \sin 4t\} = -\phi'(s) = \frac{24s^2-128}{(s^2+4^2)^3}$

Example:- Find is L.T of $\{t\, e^{-3t} \sin 2t\}$

Solution

Let $\quad f(t) = e^{-3t} \sin 2t \quad \Rightarrow \quad \phi(s) = \frac{2}{(s+3)^2+4}$ [first shifting theorem]

$\Rightarrow \quad \phi'(s) = \frac{-4(s+3)}{((s+3)^2+4)^2}$

L.T of $\{t\, e^{-3t} \sin 2t\} = -\phi'(s) = \frac{4(s+3)}{((s+3)^2+4)^2}$

Example:- What is y(t) if $Y(s) = \ln[(s+1)/(s-1)]$

Solution

From the corollary $\quad f(t) = -\frac{1}{t}[\mathrm{L \cdot T}]^{-1}\{\phi'(s)\}$

Let $\quad \phi(s) = \ln\frac{s+1}{s-1} \quad$ or $\quad \phi(s) = \ln(s+1) - \ln(s-1)$

$\phi'(s) = \frac{1}{s+1} - \frac{1}{s-1} \quad \Rightarrow \quad \mathrm{LT}^{-1}\phi'(s) = e^{-t} - e^{t}$

$\therefore \quad f(t) = -\frac{1}{t}\,[e^{-t} - e^{t}] \cdot \frac{2}{2} \quad \Rightarrow \quad f(t) = \frac{2}{t}\left[\frac{e^t - e^{-t}}{2}\right] = \frac{2 \sinh t}{t}$

HW:- What is y(t) if $Y(s) = \ln\frac{s^2-1}{s^2}$

Example:- prove that L.T of $\{t^2 f(t)\} = \phi''(s)$

Solution

Since $\quad \{t^2 f(t)\} = \{t \cdot t f(t)\} \quad$ now let $\; t f(t) = g(t)$

then from theorem $\quad$ L.T of $\{t g(t)\} = -G'(s) \quad$ but $\quad G(s) = $ L.T of $\{t f(t)\} = -\phi'(s)$

so $\quad$ L.T of $\{t^2 f(t)\} = -[-\phi'(s)]' = \phi''(s)$

<u>HW</u>: Check $\quad$ L.T of $\{t^2 \sin 4t\}$

<u>Example</u>

Solve the following variable coefficient differential equation

$$ty''(t) + 2(t-1)y'(t) + (t-2)y(t) = 0$$

<u>Solution</u>

The given differential equation can be written as

$$ty''(t) + 2ty'(t) - 2y'(t) + ty(t) - 2y(t) = 0 \qquad \cdots\cdots\cdots(1)$$

Since $\quad$ L.T of $\{tf(t)\} = -\phi'(s)$

L.T of $\{y'(t)\} =$ L.T of $\left\{\frac{dy}{dt}\right\} = s\,Y(s) - y_o$

L.T of $\{y''(t)\} = s^2 Y(s) - sy_o - y'_o$

then $\quad$ L.T of $\{ty'(t)\} = -[Y(s) + sY'(s)]$

and $\quad$ L.T of $\{ty''(t)\} = -[s^2 Y'(s) + 2sY(s) - y_o]$

now, taking L.T of both sides of equation (1)

$$-[s^2 Y'(s) + 2sY(s) - y_o] - 2[Y(s) + sY'(s)] - 2[s\,Y(s) - y_o] - Y'(s) - 2Y(s) = 0$$

after rearranging we obtain

$$-(s^2 + 2s + 1)Y'(s) - 4(s+1)Y(s) = -3y_o$$

$$(s+1)^2 Y'(s) + 4(s+1)Y(s) = -3y_o$$

or $\quad Y'(s) + \frac{4}{(s+1)} Y(s) = \frac{3y_o}{(s+1)^2} \qquad \cdots\cdots\cdots(2)$

this equation is linear first order differential equation which can be solved by **Integrating Factor**

<u>Integrating Factor</u>

The solution of differential equation of the form

$\frac{dy}{dx} + P(x)\,y = Q(x)$ has solution of the form

$$y = Ce^{-h} + e^{-h} \int e^{h}\, Q(x)dx$$

Where $\quad h = \int P(x)\,dx =$ Integrating Factor

Now comparing with equation (2)

$$P(x) = \frac{4}{(s+1)} \quad \text{and} \quad Q(x) = \frac{3y_o}{(s+1)^2}$$

So $\quad h = \int \frac{4}{(s+1)} ds = 4\ln(s+1) = \ln(s+1)^4$

$$Y(s) = Ce^{-\ln(s+1)^4} + e^{-\ln(s+1)^4} \int e^{\ln(s+1)^4}\ \frac{3y_o}{(s+1)^2} ds$$

$$Y(s) = \frac{C}{(s+1)^4} + \frac{1}{(s+1)^4}\int (s+1)^2\, 3y_o\, ds$$

$$Y(s) = \frac{C}{(s+1)^4} + \frac{y_o}{s+1}$$

The inverse L.T is $\boxed{y(t) = \frac{t^3}{3!} Ce^{-t} + y_o e^{-t}}$

2-Integration Theorem

If $f(t)$ is piecewise regular on $[0, \infty]$ and of exponential order and if L.T of $f(t) = \phi(s)$, and if $f(t)/t$ has a limit as t approaches zero from the right then:-

$$\text{L.T of}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \phi(s)\, ds$$

This theorem means that integration of the transform of a function $f(t)$ corresponds to the division of $f(t)$ by t

Prove

From the definition of L.T of $f(t)$ $\quad \text{L.T } of\ \{f(t)\} = \phi(s) = \int_0^\infty f(t)\ e^{-st}\ dt$

integration both sides of this $\quad \int_s^\infty \phi(s)\, ds = \int_s^\infty \left[\int_0^\infty f(t)\ e^{-st}\ dt\right] ds$

by reversing the order of integration $\quad \int_s^\infty \phi(s)\, ds = \int_0^\infty \int_s^\infty f(t)\ e^{-st}\, ds\, dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$

$$\int_s^\infty \phi(s)\, ds = \int_0^\infty \frac{f(t)}{t}\ e^{-st}\ dt = \text{L.T of}\left\{\frac{f(t)}{t}\right\}$$

Corollary

By taking inverse of a integration theorem

$$\frac{f(t)}{t} = \text{LT}^{-1}\int_s^\infty \phi(s)\, ds \qquad \Rightarrow \qquad f(t) = t\ \text{LT}^{-1}\int_s^\infty \phi(s)\, ds$$

This Corollary is useful in finding inverse when the integral of a transform is simpler to work with than the transform itself.

<u>Example</u>:- What is L.T of $\left\{\frac{\sin kt}{t}\right\}$

<u>Solution</u>

Let $f(t) = \sin kt \quad \Rightarrow \quad \phi(s) = \frac{k}{s^2+k^2}$

applying the integration theorem $\quad \int_s^\infty \phi(s)\, ds = \int_s^\infty \frac{k}{s^2+k^2}\, ds$

let $\quad s = k\tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}\frac{s}{k} \quad \Rightarrow$

$\quad ds = k\sec^2\theta\ d\theta$

so $\quad \int_s^\infty \frac{k}{s^2+k^2}\, ds = \int_s^\infty \frac{k^2\sec^2\theta}{k^2\tan^2\theta + k^2} d\theta$

Recall the assumptions:-

1- $a^2 + u^2$ Let $u = a\tan\theta$

2- $a^2 - u^2$ Let $u = a\sin\theta$

3- $u^2 - a^2$ Let $u = a\sec\theta$