Example:- Evaluate $\int_0^\infty \sqrt{z} e^{-z^3} dz$

Solution let $t = z^3 \quad \Rightarrow \quad z = t^{1/3}$

$$dt = 3z^2 dz \quad \Rightarrow \quad dz = \frac{1}{3} z^{-2} dt$$

then $\int_0^\infty \sqrt{z} e^{-z^3} dz = \int_0^\infty \sqrt{t^{1/3}} e^{-t} \ \frac{1}{3} t^{-\frac{2}{3}} dt \ = \ \frac{1}{3} \int_0^\infty e^{-t} \ t^{-\frac{1}{2}-1+1} dt$

$$\therefore \quad \int_0^\infty \sqrt{z} e^{-z^3} dz = \frac{1}{3} \int_0^\infty e^{-t} \ t^{\frac{1}{2}-1} dt = \frac{1}{3} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

Euler Beta Function

The Beta function $\beta(x, y)$ is defined by the integral

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \qquad x > 0 \ , \qquad y > 0$$

The Euler Beta function can be represented in terms of Gamma function as:-

$$\boxed{\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$$

Prove that $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Since $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ let $t = m^2 \quad \Rightarrow \quad dt = 2m \, dm$

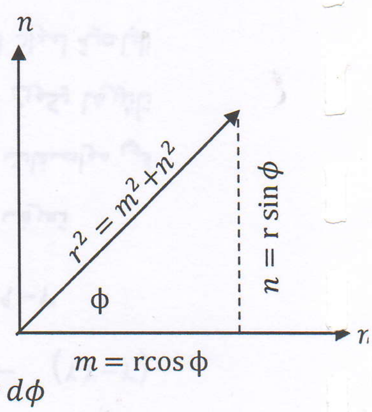
then $\Gamma(x) = \int_0^\infty e^{-m^2} m^{2x-2} 2m dm = 2 \int_0^\infty e^{-m^2} m^{2x-1} dm$

similarly $\Gamma(y) = \int_0^\infty e^{-t} t^{y-1} dt$ let $t = n^2 \quad \Rightarrow \quad dt = 2n \, dn$

then $\Gamma(y) = \int_0^\infty e^{-n^2} n^{2y-2} 2n dn = 2 \int_0^\infty e^{-n^2} n^{2y-1} dn$

multiplying $\Gamma(x) \cdot \Gamma(y)$

$$\Gamma(x) \cdot \Gamma(y) = 4 \int_0^\infty \int_0^\infty e^{-(m^2+n^2)} m^{2x-1} n^{2y-1} dn \, dm$$



By transformation this integral to the polar coordinates where

$$m = r \cos \phi \quad , \qquad n = r \sin \phi \qquad dm \, dn = r dr d\phi$$

$$\Gamma(x) \cdot \Gamma(y) = 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} (r \cos \phi)^{2x-1} (r \sin \phi)^{2y-1} r dr \, d\phi$$

since $r^{2x-1} \cdot r^{2y-1} \cdot r = r^{2(x+y)} \cdot r^{-1} = r^{2(x+y)-1}$

$$\Rightarrow \quad \Gamma(x) \cdot \Gamma(y) = 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} r^{2(x+y)-1} (\cos \phi)^{2x-1} (\sin \phi)^{2y-1} dr \, d\phi$$

Or $\Gamma(x) \cdot \Gamma(y) = 4 (\int_0^{\pi/2} (\cos \phi)^{2x-1} (\sin \phi)^{2y-1} \, d\phi) (\int_0^\infty e^{-r^2} r^{2(x+y)-1} dr)$

Now, $\int_0^{\pi/2} (\cos\phi)^{2x-1}(\sin\phi)^{2y-1}\, d\phi \qquad \ldots\ldots\ldots(a)$

let $t = \sin^2\phi$ when $\phi = 0,\ t = 0$, and when $\phi = \frac{\pi}{2}$, $t = 1$

Since $\sin^2\phi + \cos^2\phi = 1 \Rightarrow \cos\phi = (1-t)^{1/2}$

$$dt = 2\sin\phi\cos\phi\, d\phi \Rightarrow d\phi = \frac{dt}{2\sin\phi\cos\phi} = \frac{dt}{2\,t^{1/2}(1-t)^{1/2}}$$

Substitution these values into integral (a)

$$\int_0^{\pi/2} (\cos\phi)^{2x-1}(\sin\phi)^{2y-1}\, d\phi = \int_0^1 (1-t)^{y-\frac{1}{2}}(t)^{x-\frac{1}{2}} \frac{dt}{2\,t^{1/2}(1-t)^{1/2}}$$

$$\int_0^{\pi/2} (\cos\phi)^{2x-1}(\sin\phi)^{2y-1}\, d\phi = \frac{1}{2}\int_0^1 (1-t)^{y-1}(t)^{x-1}\, dt = \frac{1}{2}\beta(x,y)$$

$$\int_0^{\infty} e^{-r^2} r^{2(x+y)-1}\, dr \qquad \ldots\ldots\ldots(b)$$

Let $r^2 = t \Rightarrow 2r\,dr = dt$

so $\therefore \int_0^{\infty} e^{-t} t^{(x+y)-\frac{1}{2}} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^{\infty} e^{-t} t^{(x+y)-1}\, dt$

$$\therefore \int_0^{\infty} e^{-r^2} r^{2(x+y)-1}\, dr = \frac{1}{2}\,\Gamma(x+y)$$

$$\Gamma(x)\cdot\Gamma(y) = 4\cdot\frac{1}{2}\,\Gamma(x+y)\cdot\frac{1}{2}\beta(x,y)$$

Or $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Example:- Evaluate $\int_0^1 t^{0.3}\,(1-t)^{0.9} dt$

Solution

$$\int_0^1 t^{1.3-1}\,(1-t)^{1.9-1} dt = \beta(1.3,1.9) = \frac{\Gamma(1.3)\Gamma(1.9)}{\Gamma(1.3+1.9)} = \frac{\Gamma(1.3)\Gamma(1.9)}{\Gamma(3.2)}$$

but $1.2(1.2+1)\Gamma(1.2) = \Gamma(1.2+1+1) = \Gamma(3.2)$

so $\Gamma(3.2) = 1.2\cdot 2.2\cdot 0.9182 = 2.424$

$\therefore \int_0^1 t^{0.3}\,(1-t)^{0.9} dt = \frac{0.8975\cdot 0.962}{2.424} = 0.3562$

Example:- Evaluate $\int_0^{\infty} \frac{y^{a-1}}{1+y}\, dy$

Solution $\int_0^{\infty} \frac{y^{a-1}}{1+y}\, dy = \int_0^{\infty} y^{a-1}(1+y)^{-1}\, dy$

Let $y = \frac{x}{1-x}$ where $y = 0$ when $x = 0$ and $y \to \infty$ when $x = 1$

$$\Rightarrow \quad dy = \frac{1-x+x}{(1-x)^2}dx = \frac{dx}{(1-x)^2}$$

Then the given integral becomes to

$$\int_0^\infty \frac{y^{a-1}}{1+y}dy = \int_0^1 \left(\frac{x}{1-x}\right)^{a-1}\left(1+\frac{x}{1-x}\right)^{-1}(1-x)^{-2}dx$$

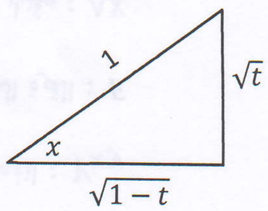
$$= \int_0^1 \left(\frac{x}{1-x}\right)^{a-1}\left(\frac{1}{1-x}\right)^{-1}(1-x)^{-2}dx = \int_0^1 x^{a-1}(1-x)^{-(a-1)}(1-x)^{-1}dx$$

$$= \int_0^1 x^{a-1}(1-x)^{-a}dx = \int_0^1 x^{a-1}(1-x)^{-a+1-1}dx$$

$$= \int_0^1 x^{a-1}(1-x)^{(1-a)-1}dx = \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)} = \Gamma(a)\Gamma(1-a)$$

Example:- Evaluate $\int_0^{\pi/2} \cos^{2m-1}x \, \sin^{2n-1}x \, dx$

Solution let $\sin x = \sqrt{t}$ ………… (1)



then from the figure $\cos x = \sqrt{1-t}$ ………… (2)

from equation (1) $\cos x \, dx = \frac{1}{2}\, t^{-1/2}\, dt$

$$\Rightarrow \quad dx = \frac{1}{2}\frac{1}{\sqrt{t}}\frac{1}{\cos x}dt \quad \Rightarrow \quad dx = \frac{1}{2}\frac{1}{\sqrt{t}}\frac{1}{\sqrt{1-t}}dt$$

Also from equation when $x = 0$, $t = 0$, and when $x = \frac{\pi}{2}$, $t = 1$

Then $\int_0^{\pi/2} \cos^{2m-1}x \, \sin^{2n-1}x \, dx = \int_0^1 (1-t)^{\left(\frac{2m-1}{2}\right)} \quad (t)^{\left(\frac{2n-1}{2}\right)} \; \frac{1}{2}\,\frac{1}{\sqrt{t}}\,\frac{1}{\sqrt{1-t}}dt$

Or $= \frac{1}{2}\int_0^1 (1-t)^{m-1}(t)^{n-1}dt$

$$\therefore \quad \int_0^{\pi/2} \cos^{2m-1}x \, \sin^{2n-1}x \, dx = \frac{1}{2}\,\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

H.WS

1- Evaluate each of the following

(a) - $\int_0^\infty (x+1)^2 \, e^{-x^3}\, dx$ (b) - $\int_0^\infty \exp(-\sqrt{x})\, dx$ (c) - $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx$

2- Evaluate each of the following

(a) $\int_0^{\pi/2} \sqrt{\cos\theta}\ \ d\theta$ **Answer** $\frac{\sqrt{\pi}}{2}\ \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}$

(b) $\int_0^{\pi/2} \sqrt{\tan\theta}\ \ d\theta$ **Answer** $\frac{1}{2}\ \Gamma\left(\frac{3}{4}\right)\ \Gamma(\frac{1}{4})$

(c) $\int_0^{1} \frac{dx}{\sqrt{\ln(1/x)}}$ let $\ln(1/x) = z$ **Answer** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(d) $\int_0^{1} x^m \left(\ln\frac{1}{x}\right)^n\ dx$ **Answer** $\frac{1}{(m+1)^{n+1}}\ \Gamma(n+1)$

(e) $\int_0^{1} \frac{dz}{\sqrt{1-z^4}}$ let $z^4 = x$ **Answer** $\frac{\sqrt{\pi}}{4}\ \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$

3-By setting $2m-1 = k$ and $n = \frac{1}{2}$ in the result of $\int_0^{\pi/2} \cos^{2m-1}x\ \ \sin^{2n-1}x\ dx$, show that

$$\int_0^{\pi/2} \cos^k\theta\ d\theta = \frac{\sqrt{\pi}}{2}\ \frac{\Gamma[(k+1)/2]}{\Gamma[(k/2)+1]}\ ,\qquad k > -1$$

What is $\int_0^{\pi/2} \sin^k\theta\ d\theta$