Orthogonal Properties of Sine and Cosine :

Definition 1:- If a sequence of real functions

$$\{\phi_n(x)\} \qquad n = 1,2,3,\cdots$$

which are defined over some interval (a,b), finite or infinite, has the property that

$$\int_a^b \phi_m(x)\phi_n(x)dx \begin{cases} = 0, & m \neq n \\ \neq 0, & m = n \end{cases}$$

Then the functions are said to form an **orthogonal** set on that interval.

Definition 2:- If the functions of an orthogonal set $\{\phi_n(x)\}$ have the property that

$$\int_a^b \phi_n^2(x)dx = 1$$

For all values of n then the functions are said to be **orthonormal** on that interval (a,b).

Notes :

1- It is no specialization to assume that an orthogonal set of functions is also orthonormal.

2- Any set of orthogonal functions can easily be converted into an orthonormal set. In fact, if the function of the set $\{\phi_n(x)\}$ are orthogonal and if k_n is the value of $\int_a^b \phi_n^2(x)dx$, then the function $\frac{\phi_1(x)}{\sqrt{k_1}}$, $\frac{\phi_2(x)}{\sqrt{k_2}}$, $\frac{\phi_3(x)}{\sqrt{k_3}}$, …… (k_n must be positive) are orthonormal.

Definition 3:- If a sequence of real functions

$$\{\phi_n(x)\} \qquad n = 1,2,3,\cdots$$

has the property that over some interval (a,b), finite or infinite

$$\int_a^b p(x)\phi_m(x)\phi_n(x)dx \begin{cases} = 0, & m \neq n \\ \neq 0, & m = n \end{cases}$$

Then the functions are said to be **orthogonal** with respect to the weight function p(x) on that interval.

Now, 1- any set of functions orthogonal with respect to a weight function p(x) can be converted into a set of functions orthogonal in the first sense (Definition 1) simply by multiplying each member of the set by $\sqrt{p(x)}$. $p(x) > 0$

2- with respect to any set of functions $\{\phi_n(x)\}$ orthogonal over an interval (a,b), an arbitrary function f(x) has a formal expansion analogous to a Fourier expansion,

$$f(x) = C_1\phi_1(x) + C_2\phi_2(x) + C_3\phi_3(x) + \cdots + C_n\phi_n(x) + \cdots \qquad \cdots\cdots\cdots(*)$$

then multiplying both sides of equation (1) by $\phi_n(x)$ and integrating formally between the appropriate limits a and b, we have

$$\int_a^b f(x)\phi_n(x)dx = a_1 \int_a^b \phi_n(x)\phi_1(x)dx + a_2 \int_a^b \phi_n(x)\phi_2(x)dx + a_3 \int_a^b \phi_n(x)\phi_3(x)dx +$$

$$+\int_a^b \phi_n(x)\phi_n(x)dx\, a_n + a_{n+1} \int_a^b \phi_n(x)\phi_{n+1}(x)dx + \cdots\cdots$$

From the property of orthogonality, all integrals on the right are zero except $\int_a^b \phi_n^2(x)dx$

$$\boxed{a_n = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi_n^2(x)dx}}$$

Example:-

Show that the given set is orthogonal on the given interval I and determine the corresponding orthonormal set

$\{ 1 , \cos x , \cos 2x , \cos 3x , \cdots\cdots , \cos nx , \cdots\cdots \}$, $0 \le x \le 2\pi$

Solution

Let $\phi_n(x) = \cos nx \quad \phi_m(x) = \cos mx$, in the $\int_a^b \phi_n(x)\phi_m(x)dx$

$$\int_0^{2\pi} \cos mx.\cos nx dx = \int_0^{2\pi} \frac{1}{2}[\cos(n+m)x + \cos(n-m)x]dx$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right]_0^{2\pi} \quad \cdots\cdots\cdots(1)$$

Since n and m are integer $\therefore$ eq. (1) = 0 for $n \neq m$

Now, for n = m

the first term in eq. (1) = 0 , but for second term we take limit as $n \to m$ as follow

$$\lim_{n \to m} \left.\frac{\sin(n-m)x}{n-m}\right]_0^{2\pi} = \lim_{n \to m} \frac{\sin(n-m).2\pi}{n-m}$$

Taking limit using L'opitals rule $\lim_{n \to m} \frac{2\pi.\cos(n-m).2\pi}{1} = 2\pi$

$$\int_0^{2\pi} \cos^2 nx.dx = \frac{1}{2}.2\pi = \boxed{\pi}$$

Now we need checking the orthogonality condition for 1 with $\cos nx$ for $n = 1,2,3,\cdots$

$$\int_0^{2\pi} 1.\cos nx dx = \left.\frac{\sin nx}{n}\right]_0^{2\pi} = 0,\ n = 1,2,3,\cdots$$

For n = 0, $\cos 0 = 1$

$$\int_0^{2\pi} 1^2.dx = 2\pi$$

$\therefore$ the given set is orthogonal on the interval $0 \le x \le 2\pi$

Then the corresponding orthonormal set is

$$\frac{1}{\sqrt{2\pi}},\ \frac{\cos x}{\sqrt{\pi}},\ \frac{\cos 2x}{\sqrt{\pi}},\ \frac{\cos 3x}{\sqrt{\pi}},\ \ldots\ldots$$

Definition 4 :- A real function f(x) is said to be **Null function** on the interval (a,b) if

$$\int_a^b f^2(x)dx = 0$$

Example:-

Show that the set $\{\ \sin nx\ \}$ are orthogonal at the interval $(-\pi, \pi)$ and then show that the function $g(x) = x^2$ cannot be represented on this interval by a series of the form

$$C_1 \sin x + C_2 \sin 2x + C_3 \sin 3x + \cdots + C_n \sin nx + \cdots$$

Solution

Let $\phi_n(x) = \sin nx$ $\phi_m(x) = \sin mx$, in the $\int_a^b \phi_n(x)\phi_m(x)dx$

$$\int_{-\pi}^{\pi} \sin mx.\sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2}[-\cos(n+m)x + \cos(n-m)x]dx$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} \quad \cdots\cdots\cdots(1)$$

Since n and m are integer $\therefore$ eq. (1) = 0 for $n \neq m$

Now, for n = m

the first term in eq. (1) = 0 , but for second term we take limit as $n \to m$ as follow

$$\lim_{n \to m} \left.\frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} = \lim_{n \to m} \frac{\sin(n-m).2\pi}{n-m} = \pi$$

$\therefore$ the given set is orthogonal on the interval $-\pi \le x \le \pi$

Then, let us find C_n

$$C_n = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi_n^2(x)dx} = \frac{\int_{-\pi}^{\pi} x^2 \sin nxdx}{\int_{-\pi}^{\pi} \sin^2 nxdx} = \frac{\int_{-\pi}^{\pi} x^2 \sin nxdx}{\pi}$$

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x^2	+ sinnx
2x	$\frac{\cos nx}{n}$
2	- $\frac{\sin nx}{n^2}$
0	+ $\frac{\cos nx}{n^3}$

Using integrating by part
Note that $\cos(n\pi) = \cos(-n\pi)$

$$C_n = \frac{\frac{2}{n^3}[\cos n\pi - \cos(-n\pi)]}{\pi} = 0$$

Partial Fraction Expansion

In many cases the solutions are usually appears as a quotient of polynomials

$$G(x) = Q(x)/P(x) \quad \cdots\cdots\cdots (1)$$

Where Q(x) and P(x) are polynomials of x. It is assumed that the order of P (x) is greater than Q(x). The polynomial P(x) may be written as

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots\cdots\cdots + a_1x + a_0$$

…given for the cases of simple pole, multiple – order poles, and complex conjugate poles of G(x)

1- G(x) has simple poles

If all the poles of G(x) are simple and real, equation (1) can be written as

$$G(x) = \frac{Q(x)}{P(x)} = \frac{Q(x)}{(x+x_1)(x+x_2)\cdots\cdots(x+x_n)} \quad \cdots\cdots\cdots (2)$$

where $x_1 \neq x_2 \neq \cdots\cdots \neq x_n$. Applying partial fraction expansion equation (2) becomes to

$$G(x) = \frac{k_1}{(x+x_1)} + \frac{k_2}{(x+x_2)} + \cdots\cdots + \frac{k_n}{(x+x_n)}$$

The coefficients k_i $(i = 1,2,3,\cdots\cdots,n)$ is determined by multiplying both sides of equation (2) by the factor $(x + x_i)$ and then letting x equal to $-x_i$ or

$$k_i = \left[(x + x_i)\frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

Example:- Expand the following by Partial Fraction $G(x) = \frac{5x+3}{x^3+6x^2+11x+6}$

Solution $G(x) = \frac{5x+3}{x^3+6x^2+11x+6} = \frac{5x+3}{(x+1)(x+2)(x+3)}$

then the Partial Fraction form of $G(x)$ is $\frac{5x+3}{(x+1)(x+2)(x+3)} = \frac{k_1}{x+1} + \frac{k_2}{x+2} + \frac{k_3}{x+3}$

to find k_1 multiply both sides by $x+1$ then let $x=-1$

$$k_1 = \frac{5(-1)+3}{(2-1)(3-1)} = -1$$

$$k_2 = \frac{5(-2)+3}{(1-2)(3-2)} = 7$$

$$k_3 = \frac{5(-3)+3}{(1-3)(2-3)} = -6$$

$$G(x) = \frac{-1}{x+1} + \frac{7}{x+2} - \frac{6}{x+3}$$

2- G(x) has multiple – order poles

If r of the n poles of $G(x)$ are identical, or we say that the pole at $x=-x_i$ is of multiplicity r, $G(x)$ is written as

$$G(x) = \frac{Q(x)}{P(x)} = \frac{Q(x)}{(x+x_1)(x+x_2)\cdots\cdots(x+x_{n-r})(x+x_i)^r} \quad , \qquad i \neq 1,2,\cdots\cdots,n-r$$

Then

$$G(x) = \underbrace{\frac{k_1}{(x+x_1)} + \frac{k_2}{(x+x_2)} + \cdots\cdots + \frac{k_{n-r}}{(x+x_{n-r})}}_{n-r \text{ terms of simple poles}} + \underbrace{\frac{A_1}{(x+x_i)} + \frac{A_2}{(x+x_i)^2} + \cdots\cdots + \frac{A_r}{(x+x_i)^r}}_{r-\text{terms of repeated poles}}$$

Where

$$A_r = \left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

$$A_{r-1} = \frac{1}{1!}\frac{d}{dx}\left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

$$A_{r-2} = \frac{1}{2!}\frac{d^2}{dx^2}\left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A_1 = \frac{1}{(r-1)!}\frac{d^{r-1}}{dx^{r-1}}\left[(x+x_i)^r \frac{Q(x)}{P(x)}\right|_{x=-x_i}$$

<u>**Example:-**</u> Expand the following function by Partial Fraction $G(x) = \frac{1}{x(x+1)^3(x+2)}$

<u>Solution</u>

$$G(x) = \frac{1}{x(x+1)^3(x+2)} = \frac{k_1}{x} + \frac{k_2}{x+2} + \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3}$$

then $k_1 = \frac{1}{(1^3)(2)} = \frac{1}{2}$

$$k_2 = \frac{1}{(-2)(-2+1)^3} = \frac{1}{2}$$

$$A_3 = [(x+1)^3 G(x)|_{x=-1} = -1$$

$$A_2 = \frac{d}{dx}[(x+1)^3 G(x)|_{x=-1} = \frac{d}{dx}\left[\frac{1}{x(x+2)}\right|_{x=-1} = -\left[\frac{2x+2}{x^2(x+2)^2}\right|_{x=-1} = 0$$

$$A_1 = \frac{1}{2!}\frac{d^2}{dx^2}[(x+1)^3 G(x)\Big|_{x=-1} = \frac{1}{2!}\frac{d^2}{dx^2}\left[\frac{1}{x(x+2)}\right|_{x=-1} = -1$$

Substituting these values $G(x) = \frac{1}{2x} + \frac{1}{2(x+2)} - \frac{1}{x+1} - \frac{1}{(x+1)^3}$

3- G(x) has simple complex – conjugate poles

Suppose that P(x) has simple complex conjugate poles with α_1 as real part and α_2 as imaginary part then

$$G(x) = \frac{Q(x)}{P(x)} = \frac{Q(x)}{(x+\alpha_1-i\alpha_2)(x+\alpha_1+i\alpha_2)}$$

The expansion by partial fraction gives

$$G(x) = \frac{k_{-\alpha_1+i\alpha_2}}{x+\alpha_1-i\alpha_2} + \frac{k_{-\alpha_1-i\alpha_2}}{x+\alpha_1+i\alpha_2}$$

where $k_{-\alpha_1+i\alpha_2} = (x+\alpha_1-i\alpha_2)\, G(x)|_{x=-\alpha_1+i\alpha_2}$

and $k_{-\alpha_1-i\alpha_2} = (x+\alpha_1+i\alpha_2)\, G(x)|_{x=-\alpha_1-i\alpha_2}$

<u>**Example:-**</u> Expand the following function by Partial Fraction $G(x) = \frac{x+2}{(x+1)(x^2+4)}$

<u>Solution</u> $G(x) = \frac{x+2}{(x+1)(x+2i)(x-2i)} = \frac{k_1}{(x+1)} + \frac{k_{-0-2i}}{(x+2i)} + \frac{k_{-0+2i}}{(x-2i)}$

where $k_1 = \frac{x+2}{(x^2+4)}\Big|_{x=-1} = \frac{1}{5}$

$$k_{-0-2i} = \frac{x+2}{(x+1)(x-2i)}\Big|_{x=-0-2i} = \frac{2-2i}{-8-4i}\cdot\frac{-8+4i}{-8+4i} = \frac{24i-8}{80}$$

$$k_{-0+2i} = \frac{x+2}{(x+1)(x+2i)}\Big|_{x=-0+2i} = \frac{2+2i}{-8+4i}\cdot\frac{-8-4i}{-8-4i} = \frac{-24i-8}{80}$$