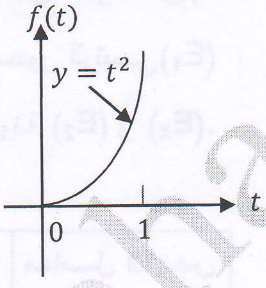
Collectively, we speak of such series as **half-range expansion**

Example:- Find the Fourier coefficients in the half-range sine expansion of the function

$$f(t) = t^2 \qquad 0 \le t < 1$$



Solution

From the graph $\quad p = 1$

The half-range sine expansion of the function is the Fourier series of the odd function ,then

$$a_n = 0 \qquad b_n = \frac{2}{p}\int_0^p f(t)\sin\frac{n\pi t}{p}\,dt$$

Or $\quad b_n = \frac{2}{1}\int_0^1 t^2 \sin\frac{n\pi t}{1}\,dt =$

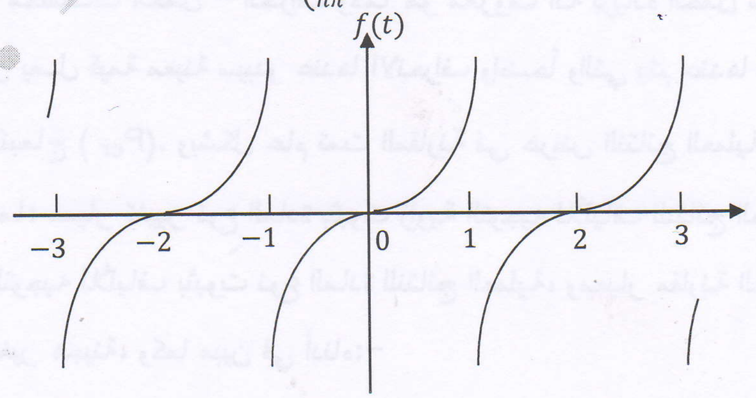
D		I
t^2	+	$\sin n\pi t$
$2t$	−	$-\frac{1}{n\pi}\cos n\pi t$
2	+	$-\left(\frac{1}{n\pi}\right)^2 \sin n\pi t$
0		$\left(\frac{1}{n\pi}\right)^3 \cos n\pi t$

$$b_n = 2\left[-\frac{t^2}{n\pi}\cos n\pi t + 2t\left(\frac{1}{n\pi}\right)^2 \sin n\pi t + 2\left(\frac{1}{n\pi}\right)^3 \cos n\pi t\right]_0^1$$

$$b_n = -\frac{2}{n\pi}\cos n\pi + 4\left(\frac{1}{n\pi}\right)^3 [\cos n\pi - 1]$$

since $\quad \cos n\pi = (-1)^n$

$$b_n = \frac{2(-1)^{n+1}}{n\pi} - \frac{4}{n^3\pi^3}[(-1)^{n+1} + 1] = \begin{cases} \frac{2}{n\pi} - \frac{8}{n^3\pi^3} & n\ odd \\ \frac{-2}{n\pi} & n\ even \end{cases}$$



The half-range cosine expansion of the function is the Fourier series of the even function ,then

$$b_n = 0 \qquad a_n = \frac{2}{p}\int_0^p f(t)\,\cos\frac{n\pi t}{p}\,dt$$

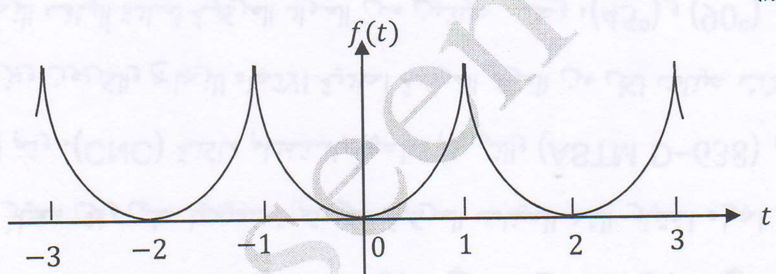
Or $\quad a_n = \frac{2}{1}\int_0^1 t^2 \cos n\pi t\; dt$

$$a_n = 2\left[\frac{t^2}{n\pi}\sin n\pi t + 2t\left(\frac{1}{n\pi}\right)^2 \cos n\pi t - 2\left(\frac{1}{n\pi}\right)^3 \sin n\pi t\right]_0^1$$

D		I
t^2	$+$	$\cos n\pi t$
$2t$	$-$	$\frac{1}{n\pi}\sin n\pi t$
2	$+$	$-\left(\frac{1}{n\pi}\right)^2 \cos n\pi t$
0		$-\left(\frac{1}{n\pi}\right)^3 \sin n\pi t$

$$a_n = \frac{2^2}{n^2\pi^2}\cos n\pi = \frac{2^2}{n^2\pi^2}(-1)^n \qquad n \neq 0$$

for $\quad n = 0 \qquad a_0 = \frac{2}{1}\int_0^1 t^2\; dt = \frac{2}{3}$



PROBLMES

Find the half-range cosine and sine expansion of each of the following functions.

1- $f(t) = \begin{cases} 1 & 0 \le t \le \pi \\ 0 & \pi < t \le 2\pi \end{cases}$

2- $f(t) = \begin{cases} at & 0 \le t \le l/2 \\ a(l-t) & l/2 < t \le l \end{cases}$

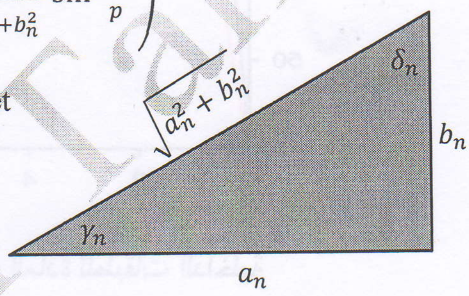
Alternative Form of Fourier Series

The standard form of Fourier series is

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}\right) \qquad \cdots\cdots\cdots (1)$$

we can apply to each pair of terms of the same frequency the usual procedure for reducing the sum of a sine and a cosine of the same angle to a single term:

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\left(\frac{a_n}{\sqrt{a_n^2+b_n^2}} \cos\frac{n\pi t}{p} + \frac{b_n}{\sqrt{a_n^2+b_n^2}} \sin\frac{n\pi t}{p}\right)$$



If we now define the angle γ_n and δ_n from the triangle shown and set

$$A_o = \frac{a_o}{2} \qquad \text{and} \qquad A_n = \sqrt{a_n^2 + b_n^2}$$

$$f(t) = A_o + \sum_{n=1}^{\infty} A_n \left(\cos\frac{n\pi t}{p} \cos\gamma_n + \sin\frac{n\pi t}{p} \sin\gamma_n\right)$$

Since $\cos x \cos y + \sin x \sin y = \cos(x - y)$

Then $\boxed{f(t) = A_o + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{p} - \gamma_n\right)}$

Similarly

$$f(t) = A_o + \sum_{n=1}^{\infty} A_n \left(\cos\frac{n\pi t}{p} \sin\delta_n + \sin\frac{n\pi t}{p} \cos\delta_n\right)$$

Since $\cos x \sin y + \sin x \cos y = \cos(x + y)$

Then $\boxed{f(t) = A_o + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi t}{p} + \delta_n\right)}$

Where $A_n = \sqrt{a_n^2 + b_n^2}$ = amplitude of the n-th harmonic

Now, since

$$\cos\frac{n\pi t}{p} = \frac{e^{ni\pi t/p} + e^{-ni\pi t/p}}{2} \qquad \sin\frac{n\pi t}{p} = \frac{e^{ni\pi t/p} - e^{-ni\pi t/p}}{2i}$$

The standard Fourier series (1) can be written as

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{ni\pi t/p} + e^{-ni\pi t/p}}{2} + b_n \frac{e^{ni\pi t/p} - e^{-ni\pi t/p}}{2i}\right)$$

Collecting terms on the various exponentials and noting that $1/i = -i$, we obtain

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2} e^{ni\pi t/p} + \frac{a_n + ib_n}{2} e^{-ni\pi t/p}\right)$$

Let $c_o = \frac{a_o}{2}$ $\quad c_n = \frac{a_n - ib_n}{2}$ $\quad c_{-n} = \frac{a_n + ib_n}{2}$

$$f(t) = c_o + \sum_{n=1}^{\infty}\left(c_n\, e^{ni\pi t/p} + c_{-n}\, e^{-ni\pi t/p}\right) \qquad \cdots\cdots\cdots (2)$$

The equation (2) can be written as

$$f(t) = \sum_{-\infty}^{\infty} c_n\, e^{ni\pi t/p}$$ **[complex exponential Fourier series]**

To find the coefficients c_o, c_n, and c_{-n}

$$c_o = \frac{a_o}{2} = \frac{1}{2p}\int_d^{d+2p} f(t)\, dt \qquad \cdots\cdots\cdots (a)$$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_d^{d+2p} f(t)\cos\frac{n\pi t}{p}\, dt - i\,\frac{1}{p}\int_d^{d+2p} f(t)\sin\frac{n\pi t}{p}\, dt\right]$$

$$c_n = \frac{1}{2p}\int_d^{d+2p} f(t)\left(\cos\frac{n\pi t}{p} - i\sin\frac{n\pi t}{p}\right) dt$$

$$c_n = \frac{1}{2p}\int_d^{d+2p} f(t)\, e^{-ni\pi t/p}\, dt \qquad \cdots\cdots\cdots (b)$$

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2}\left[\frac{1}{p}\int_d^{d+2p} f(t)\cos\frac{n\pi t}{p}\, dt + i\,\frac{1}{p}\int_d^{d+2p} f(t)\sin\frac{n\pi t}{p}\, dt\right]$$

$$c_{-n} = \frac{1}{2p}\int_d^{d+2p} f(t)\left(\cos\frac{n\pi t}{p} + i\sin\frac{n\pi t}{p}\right) dt$$

$$c_{-n} = \frac{1}{2p}\int_d^{d+2p} f(t)\, e^{ni\pi t/p}\, dt \qquad \cdots\cdots\cdots (c)$$

We see from equations (a) , (b), and (c), whether the index n is positive, negative, or zero, c_n is correctly given by the single formula

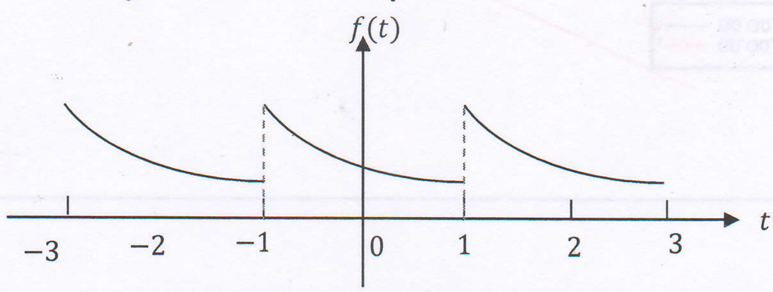
$$\boxed{c_n = \frac{1}{2p}\int_d^{d+2p} f(t)\, e^{-ni\pi t/p}\, dt}$$

Example:- Find the complex form of the Fourier series of the periodic whose definition in one period is function

$$f(t) = e^{-t} \qquad -1 < t < 1$$

Solution

From the graph $\quad 2p = 2 \quad \Rightarrow \quad p = 1$



$$c_n = \frac{1}{2p}\int_d^{d+2p} f(t)\, e^{-ni\pi t/p}\, dt = \frac{1}{2}\int_{-1}^{1} e^{-t}\, e^{-in\pi t}\, dt$$

$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)t}\, dt = \frac{1}{2}\left[-\frac{e^{-(1+in\pi)t}}{1+in\pi}\right]_{-1}^{1} = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$c_n = \frac{1}{2}\frac{e^{1}e^{in\pi} - e^{-1}e^{-in\pi}}{(1+in\pi)}$$

since $\quad e^{in\pi} = \cos n\pi + i\sin n\pi = (-1)^n$

and $\quad e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n$

then $\quad c_n = \frac{(-1)^n}{(1+in\pi)}\;\frac{e^{1} - e^{-1}}{2} = \frac{(-1)^n}{(1+in\pi)}\sinh 1 \cdot \frac{1-in\pi}{1-in\pi}$

$$c_n = \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1$$

So the complex form of Fourier series of $f(t) = e^{-t}$ is

$$e^{-t} = \sum_{-\infty}^{\infty} c_n\, e^{ni\pi t/p} = \sum_{-\infty}^{\infty} \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1 \quad e^{ni\pi t} \quad \dots\dots\dots (1)$$

Example:- Convert the exponential series (1) of previous Example into a real trigonometric form.

Solution

From the relations $\quad c_o = \frac{a_o}{2} \qquad c_n = \frac{a_n - ib_n}{2} \qquad c_{-n} = \frac{a_n + ib_n}{2}$

We see that $\quad a_o = 2c_o$

Then adding and next subtracting the expressions for c_n and c_{-n} we find that

$$a_n = c_n + c_{-n} \qquad\qquad b_n = i(c_n - c_{-n})$$

Then using $c_n = \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1$ we get

$$a_o = 2\sinh 1$$

$$a_n = \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1 + \frac{(-1)^n(1+in\pi)}{1+n^2\pi^2}\sinh 1$$ remember that $(-1)^n = (-1)^{-n}$

$$a_n = \frac{(-1)^n\sinh 1}{1+n^2\pi^2}\left[1 - ni\pi + 1 + ni\pi\right] = \frac{(-1)^n 2\sinh 1}{1+n^2\pi^2}$$

$$b_n = i\left[\frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\sinh 1 - \frac{(-1)^n(1+in\pi)}{1+n^2\pi^2}\sinh 1\right] = \frac{i\,(-1)^n\sinh 1}{1+n^2\pi^2}\left[1 - ni\pi - 1 - ni\pi\right]$$

$$b_n = \frac{(-1)^n 2\, n\pi\sinh 1}{1+n^2\pi^2}$$

1- $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 4 \end{cases}$

Answers: $A_n = \frac{1}{n\pi}\sqrt{2\left(1 - \cos\frac{n\pi}{2}\right)}$ $\quad \gamma_n = \frac{n\pi}{4}$ $\quad \delta_n = \frac{\pi}{2} - \gamma_n$

2- $f(t) = \begin{cases} 0 & -p < t < 0 \\ e^{-t} & 0 < t < p \end{cases}$

Answers: $A_n = \frac{1 - e^{-p}\cos n\pi}{\sqrt{n^2\pi^2 + p^2}}$ $\quad \gamma_n = -\tan\frac{n\pi}{p}$ $\quad \delta_n = \frac{\pi}{2} - \gamma_n$

3- $f(t) = t + t^2 \qquad -1 < t < 1$

Answers: $A_0 = \frac{1}{3}$ $\quad A_n = \frac{2}{n^2\pi^2}\sqrt{n^2\pi^2 + 4}$

Find the complex exponential Fourier series of the periodic functions whose definitions in one period are

4- $f(t) = \sin t \qquad 0 \le t \le \pi$ **Answers:** $c_n = \frac{2}{(1-4n^2)\pi}$

5- $f(t) = \sinh t \qquad -1 < t < 1$

6- $f(t) = \cosh t \qquad -1 \le t \le 1$ **Answers:** $c_n = \frac{2}{1+n^2\pi^2}$