Advanced Engineering Analysis

References 1- Advanced Engineering Mathematics by C. RAY WYLIE

2- Advanced Engineering Mathematics by ERIN KREYS

https://classroom.google.com/c/NjI1ODQwMDE5ODM3?cjc=stmlmg5 (رمز الصف الإلكتروني)

The General Linear Second – Order Differential Equation

The general linear ordinary differential equation of the second order can be written in the standard form

$$a(x)y'' + b(x)y' + c(x)y = f(x) \qquad \cdots\cdots\cdots (1)$$

Where $y'' = \frac{d^2y}{dx^2}$ and $y' = \frac{dy}{dx}$ y = dependent variable, and x = independent variable

The $a(x),\ b(x)\ and\ c(x)$ are coefficient. Equation (1) is said to be **nonhomogeneous.** If $f(x)$ is identically zero, we have the so-called **homogeneous** equation.

$$a(x)y'' + b(x)y' + c(x)y = 0 \ \textbf{(homogeneous)}$$

The Homogeneous Linear Second – Order Differential Equation with Constant Coefficients

When $a(x),\ b(x)\ and\ c(x)$ are constants the general linear second order differential equation can be written in the standard form

$$ay'' + by' + cy = 0 \qquad \cdots\cdots\cdots (2)$$

Let the solution of equation (2) be in the form

$y = e^{mx}$ where m is a constant to be determined.

Substituting $y' = me^{mx}$ and $y'' = m^2e^{mx}$ in to equation (2)

$$e^{mx}(am^2 + bm + c) = 0$$

Since $e^{mx} \neq 0$ then $\boxed{am^2 + bm + c = 0}$ ⇒ **(characteristic or auxiliary equation)**

The roots of characteristic equation can be obtained by

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, if $b^2 - 4ac = 0$ then $m_1 = m_2$ Real and equal Complete solution of the equation (2) is

$$\boxed{y(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

if $b^2 - 4ac > 0$ then $m_1 \neq m_2$ Real and unequal Complete solution of the equation (2) is

$$\boxed{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

and if

$b^2 - 4ac < 0$ then m_1 *and* m_2 are complex conjugate $\begin{matrix} m_1 = p + iq \\ m_2 = p - iq \end{matrix}$ Complete solution of the equation (2) is

$$\boxed{y(x) = e^{px}(A\cos qx + B\sin qx)}$$

Example Find complete solution of the equation $y'' + 7y' + 12y = 0$

Solution

let $y = e^{mx}$ then from given equation $e^{mx}(m^2 + 7m + 12) = 0$

then the characteristic equation becomes to $m^2 + 7m + 12 = 0$

$a = 1$ $b = 7$ $c = 12$

And its roots are $m_1 = -3$ $m_2 = -4$

Since $m_1 \neq m_2$ Real and unequal, a complete solution is

$$y(x) = c_1 e^{-3x} + c_2 e^{-4x}$$

Example Find complete solution of the equation $y'' + 2y' + 5y = 0$

Solution

let $y = e^{mx}$ then from given equation $e^{mx}(m^2 + 2m + 5) = 0$

the characteristic equation in this case is $m^2 + 2m + 5 = 0$

$a = 1$ $b = 2$ $c = 5$

And its roots are $m_1 = -1 + 2i$ $m_2 = -1 - 2i$

Since m_1 *and* m_2 are complex conjugate it clear that $p = -1$ $q = 2$

Then, a complete solution of equation is

$$y(x) = e^{-1x}(A\cos 2x + B\sin 2x)$$

Operator Notation

By definition $y' = \frac{dy}{dx} = Dy$

Then by repetition of the process of differentiation

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2y$$

Similarly $y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = D(D^2y) = D^3y$

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The operator D can be handled in many respects as though it were a simple algebraic quantity.

Particular Solution

In particular solutions, the two arbitrary constants in the complete solution must usually be determined to fit given initial (or boundary) conditions on y and y'.

Example Find the solution of the equation $y'' - 4y' + 4y = 0$ for which $y = 3$ and $y' = 4$ when $x = 0$

Solution

Using operator notation, the given $D^2y - 4Dy + 4y = 0 \Rightarrow (D^2 - 4D + 4)y = 0$

Replace D by m the characteristic equation of given equation becomes to $m^2 - 4m + 4 = 0$ and its roots are $m_1 = m_2 = 2$; hence the complete solution is

$$y(x) = c_1e^{2x} + c_2xe^{2x}$$

Now to find the particular solution, differentiate $y(x)$ with respect to x

$$y'(x) = 2c_1e^{2x} + c_2e^{2x} + 2c_2xe^{2x} = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x} =$$

The substituting the given conditions into the equations for y and y', respectively, we have

$$3 = c_1 \quad \text{and} \quad 4 = 2c_1 + 4c_2$$

Hence, $c_1 = 3$, and $c_2 = -2$

Then the required solution is $y(x) = 3e^{2x} - 2xe^{2x}$

H.WS

1- Find a complete solution of each of the following equations:

a- $y'' + 5y' = 0$ **Answer** $y(x) = c_1 + c_2e^{-5x}$

b- $(9D^2 - 12D + 4)y = 0$ **Answer** $y(x) = c_1e^{2x/3} + c_2xe^{2x/3}$

c- $y'' + 10y' + 26y = 0$ **Answer** $y(x) = e^{-5x}(A\cos x + B\sin x)$

2- Find a particular solution of each of the following equations which satisfies the given conditions:

a- $25y'' + 20y' + 4y = 0$, $y = y' = 0$ when $x = 0$
Answer $y(x) = 0$

b- $y'' + 4y = 0$, $y = 2$, when $x = 0$, $y' = 6$, when $x = 0$
Answer $y(x) = 2\cos 2x + 3\sin 2x$

The Nonhomogeneous Second – Order Differential Equation with Constant Coefficients

Consider the following nonhomogeneous Equation

$$a(x)y'' + b(x)y' + c(x)y = f(x) \qquad \cdots\cdots\cdots (1)$$

Dividing both sides of equation (1) by $a(x)$ the equation reduced to

$$y'' + p(x)y' + q(x)y = r(x) \qquad \cdots\cdots\cdots (2)$$

The solution of nonhomogeneous equation (2) is

$$y(x) = y_h + y_p$$

Where $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogeneous ODE (2) and

y_p is a particular solution of (2)

There are two methods to find particular solution y_p.

1- **Undetermined Coefficients**
2- **Variation of Parameter**

Particular Solutions by the Method of Variation of Parameter

Let $y_1(x)$, $y_2(x)$ be two homogeneous solutions of equation (2) and y_p be a particular solution of (2)

The fundamental idea behind the process is this. Instead of using two arbitrary constants c_1 and c_2 to combined two independent solutions of the homogeneous equation (2)

$$y'' + p(x)y' + q(x)y = 0$$

as we do in constructing the homogeneous solutions, we attempt to find two functions of x, say u_1, and u_2, such that

$$y_p = u_1\, y_1 + u_2y_2$$

By differentiation, $y_p' = (u_1y_1' + u_1'y_1) + (u_2y_2' + u_2'y_2) = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$

For simplicity, let $u_1'y_1 + u_2'y_2 = 0 \qquad \cdots\cdots\cdots (3)$

$$\therefore y_p' = u_1y_1' + u_2y_2'$$

and $y_p'' = (u_1y_1'' + u_1'y_1') + (u_2y_2'' + u_2'y_2')$

substituting y_p, y_p' , and y_p'' into equation we obtain

$$(u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2') + p(x)(y_1' + u_2y_2') + q(x)(u_1\, y_1 + u_2y_2) = r(x)$$

or $$u_1\underbrace{[y_1''+p(x)y_1'+q(x)y_1]}_{zero}+u_2\underbrace{[y_2''+p(x)y_2'+q(x)y_2]}_{zero}+u_1'y_1+u_2'y_2=r(x)$$

$$u_1'y_1'+u_2'y_2'=r(x) \qquad \dots\dots\dots\dots (4)$$

Equations (3) and (4) can be written in matrix form

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}=\begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

Then by grammar rule

$$u_1'=-\frac{y_2}{y_1y_2'-y_2y_1'}r(x) \quad \text{and} \quad u_2'=\frac{y_1}{y_1y_2'-y_2y_1'}r(x)$$

Since $u_1=-\int\frac{y_2r(x)}{y_1y_2'-y_2y_1'}dx$ and $u_2=\int\frac{y_1r(x)}{y_1y_2'-y_2y_1'}dx$

$$\boxed{y_p=y_2\int\frac{y_1r(x)}{y_1y_2'-y_2y_1'}dx-y_1\int\frac{y_2r(x)}{y_1y_2'-y_2y_1'}dx}$$ where $W(x)=y_1y_2'-y_2y_1'$ = Wronskian

or $$y_p=y_1\int\frac{W_1r(x)}{W}dx+y_2\int\frac{W_2r(x)}{W}dx$$

Example Find a complete solution of the equation $y''+y=\sec x$

Solution By inspection $y_h=c_1cosx+c_2sinx$

$$y_1=cosx \qquad y_2=sinx$$

$$y_1'=-\sin x \qquad y_2'=\cos x$$

then $y_1y_2'-y_2y_1'=cos^2x+sin^2x=1$, $r(x)=\sec x$

$$\therefore \quad y_p=\sin x\int cosx\,secx\,dx-cosx\int sinx\,secx\,dx=xsinx-cosx\int\frac{\sin x}{\cos x}dx$$

$$y_p=xsinx+cosx\;ln|cosx|$$

Finally $$y(x)=c_1cosx+c_2sinx+xsinx-cosx\;ln|cosx|$$

H.WS

Find a complete solution of each of the following equations:

a- $y''-y=coshx$ **Answer** $y(x)=c_1\cosh x+c_2sinhx+\frac{x\,sinhx}{2}$

b- $y''+2y'+y=e^{-x}lnx$ **Answer** $y(x)=c_1e^{-x}+c_2xe^{-x}+\left(\frac{x^2}{2}lnx-\frac{3}{4}x^2\right)e^{-x}$