Fourier Series and Fourier Integral

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Definition A function f is **periodic** if and only if if there exists a positive number $2p$ such that for every t in the domain of f, $f(t+2p)=f(t)$. The number $2p$ is called a **period** of f

Notes:-

1- If $f(t)$ and $g(t)$ have the period $2p$, then the function $h(t)=af(t)+bg(t)$.has the period $2p$ also.

2- $f(t)=$ constant is periodic.

The Euler Coefficients

Let $f(t)$ be an arbitrary periodic function of period $2p$, then $f(t)$ has formal expansion of the form

$$f(t)=\frac{1}{2}a_o+a_1\cos\frac{\pi t}{p}+a_2\cos\frac{2\pi t}{p}+\cdots\cdots\cdots+a_n\cos\frac{n\pi t}{p}+\cdots\cdots+b_1\sin\frac{\pi t}{p}+ b_2\sin\frac{2\pi t}{p}+\cdots\cdots+b_n\sin\frac{n\pi t}{p}+\cdots \qquad \cdots\cdots\cdots(1)$$

The introduction of the factor $\frac{1}{2}$ is a conventional device to render more symmetric the final formulas for the coefficients.

To determine the coefficients a_o , a_n and b_n , we need the following definite integrals, which are valid for values of d

1- $\int_d^{d+2p}\cos\frac{n\pi t}{p}dt=0 \qquad n\neq 0$

2- $\int_d^{d+2p}\sin\frac{n\pi t}{p}dt=0$

3- $\int_d^{d+2p}\cos\frac{n\pi t}{p}\cos\frac{m\pi t}{p}dt=0 \qquad n\neq m$

4- $\int_d^{d+2p}\cos^2\frac{n\pi t}{p}dt=p \qquad n\neq 0$

5- $\int_d^{d+2p}\cos\frac{m\pi t}{p}\sin\frac{n\pi t}{p}dt=0$

6- $\int_d^{d+2p}\sin\frac{m\pi t}{p}\sin\frac{n\pi t}{p}dt=0 \qquad n\neq m$

7- $\int_d^{d+2p}\sin^2\frac{n\pi t}{p}dt=p \qquad n\neq 0$

Now, to find a_o integrate both sides of equation (1) from $t=d$ to $t=d+2p$

$$\int_{d}^{d+2p} f(t)\,dt = \frac{a_o}{2}\int_{d}^{d+2p} dt + a_1\int_{d}^{d+2p} \cos\frac{\pi t}{p}\,dt + \cdots + a_n\int_{d}^{d+2p} \cos\frac{n\pi t}{p}\,dt + \cdots + b_1\int_{d}^{d+2p} \sin\frac{\pi t}{p}\,dt$$

$$+ \cdots + b_n\int_{d}^{d+2p} \sin\frac{n\pi t}{p}\,dt + \cdots$$

The first term on the right hand side is simply $= \frac{1}{2}a_o t|_d^{d+2p} = pa_o$, and by equations (2) and (3) all integrals contains cosine and sine vanishes ,then:-

$$\boxed{a_o = \frac{1}{p}\int_d^{d+2p} f(t)\,dt}$$

To find a_n $(n = 1, 2, 3, \cdots)$ multiply both sides of equation (1) by $\cos\frac{n\pi t}{p}$ and then integrate from $t = d$ to $t = d + 2p$

$$\int_{d}^{d+2p} f(t)\cos\frac{n\pi t}{p}\,dt$$

$$= \frac{a_o}{2}\int_{d}^{d+2p} \cos\frac{n\pi t}{p}\,dt + a_1\int_{d}^{d+2p} \cos\frac{\pi t}{p}\cos\frac{n\pi t}{p}\,dt + \cdots + a_n\int_{d}^{d+2p} \cos^2\frac{n\pi t}{p}\,dt + \cdots$$

$$+ b_1\int_{d}^{d+2p} \cos\frac{n\pi t}{p}\sin\frac{\pi t}{p}\,dt + \cdots + b_n\int_{d}^{d+2p} \cos\frac{n\pi t}{p}\sin\frac{n\pi t}{p}\,dt + \cdots$$

By equations (2), (4) and (6) all terms on the right hand vanishes except the one involving $\cos^2\frac{n\pi t}{p}$

$$\int_d^{d+2p} f(t)\cos\frac{n\pi t}{p}\,dt = pa_n$$

$$\boxed{a_n = \frac{1}{p}\int_d^{d+2p} f(t)\cos\frac{n\pi t}{p}\,dt}$$

To find b_n $(n = 1, 2, 3, \cdots)$ multiply both sides of equation (1) by $\sin\frac{n\pi t}{p}$ and then integrate from $t = d$ to $t = d + 2p$

$$\int_{d}^{d+2p} f(t)\sin\frac{n\pi t}{p}dt$$

$$= \frac{a_o}{2}\int_{d}^{d+2p} \sin\frac{n\pi t}{p}dt + a_1\int_{d}^{d+2p} \cos\frac{\pi t}{p}\sin\frac{n\pi t}{p}dt + \cdots + a_n\int_{d}^{d+2p} \sin\frac{n\pi t}{p}\cos\frac{n\pi t}{p}dt$$

$$+ \cdots + b_1\int_{d}^{d+2p} \sin\frac{\pi t}{p}\sin\frac{n\pi t}{p}dt + \cdots + b_n\int_{d}^{d+2p} \sin^2\frac{n\pi t}{p}dt + \cdots$$

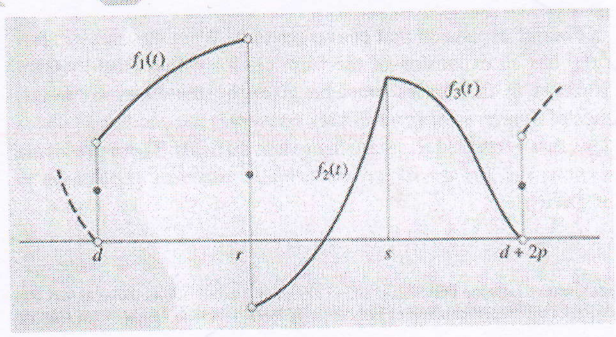
Similarly

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\frac{n\pi t}{p}dt$$

a_o , a_n , b_n are called **Euler-Fourier** formulas, and the series (1) when its coefficients have these values is known the **Fourier Series of $f(t)$**

Dirichlet Theorem If $f(t)$ is a bounded periodic function which in any one periodic has at most a finite number of local maximum and minimum and a finite number of points of discontinuity, then the Fourier series of $f(t)$ converges to $f(t)$ at all points when $f(t)$ is continuous and converges to the average of the right- and left- hand limits of $f(t)$ at each point where $f(t)$ is discontinuous.

In Figure below the function $f(t)$ is defined by three different expressions $f_1(t)$, $f_2(t)$,and $f_3(t)$ successive portions of the period interval $d \le t \le t + 2p$. Hence the Euler formulas can be written as



$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\frac{n\pi t}{p}dt =$$

$$\frac{1}{p}\int_{d}^{r} f_1(t)\cos\frac{n\pi t}{p}dt + \frac{1}{p}\int_{r}^{s} f_2(t)\cos\frac{n\pi t}{p}dt + \frac{1}{p}\int_{s}^{d+2p} f_3(t)\cos\frac{n\pi t}{p}dt$$

$$b_n = \frac{1}{p}\int_d^{d+2p} f(t)\sin\frac{n\pi t}{p}\,dt =$$

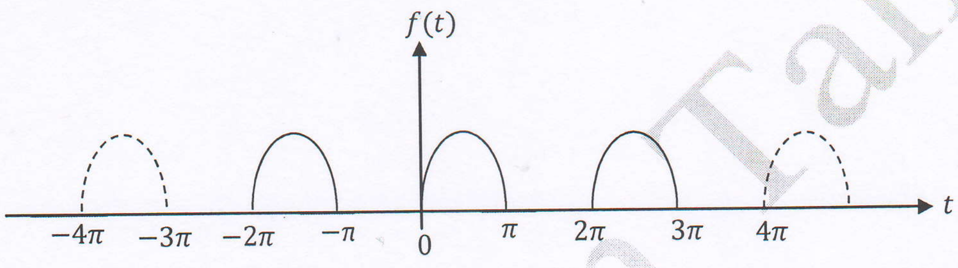
$$\frac{1}{p}\int_d^{r} f_1(t)\sin\frac{n\pi t}{p}\,dt + \frac{1}{p}\int_r^{s} f_2(t)\sin\frac{n\pi t}{p}\,dt + \frac{1}{p}\int_s^{d+2p} f_3(t)\sin\frac{n\pi t}{p}\,dt$$

<u>Example</u>:- What is the Fourier expansion of the periodic function whose definition in one period is

$$f(t) = \begin{cases} 0 & -\pi \le t \le 0 \\ \sin t & 0 \le t \le \pi \end{cases}$$

<u>Solution</u>

Graph the given function, then from the graph the half-period of the given function is $p = \pi$, taking $d = -\pi$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}\sin t\,\cos nt\,dt$$

From identities

$$\sin t\cos nt = \frac{1}{2}[\sin(1+n)t + \sin(1-n)t]$$

$$a_n = \frac{1}{\pi}\left[-\frac{1}{2}\left\{\frac{\cos(1-n)t}{1-n} + \frac{\cos(1+n)t}{1+n}\right\}\right]_0^{\pi} =$$

$$a_n = -\frac{1}{2\pi}\left[\frac{\cos(\pi-n\pi)}{1-n} + \frac{\cos(\pi+n\pi)}{1+n} - \left(\frac{1}{1-n} + \frac{1}{1+n}\right)\right] = -\frac{1}{2\pi}\left[\frac{-\cos n\pi}{1-n} + \frac{-\cos n\pi}{1+n} - \frac{2}{1-n^2}\right]$$

$$a_n = \frac{1+\cos n\pi}{\pi(1-n^2)} \qquad n \ne 1$$

For $n = 1$ $\quad a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos t\,dt = \frac{1}{\pi}\int_0^{\pi}\sin t\,\cos t\,dt = \left.\frac{\sin^2 t}{2\pi}\right|_0^{\pi} = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}\sin t\,\sin nt\,dt$$

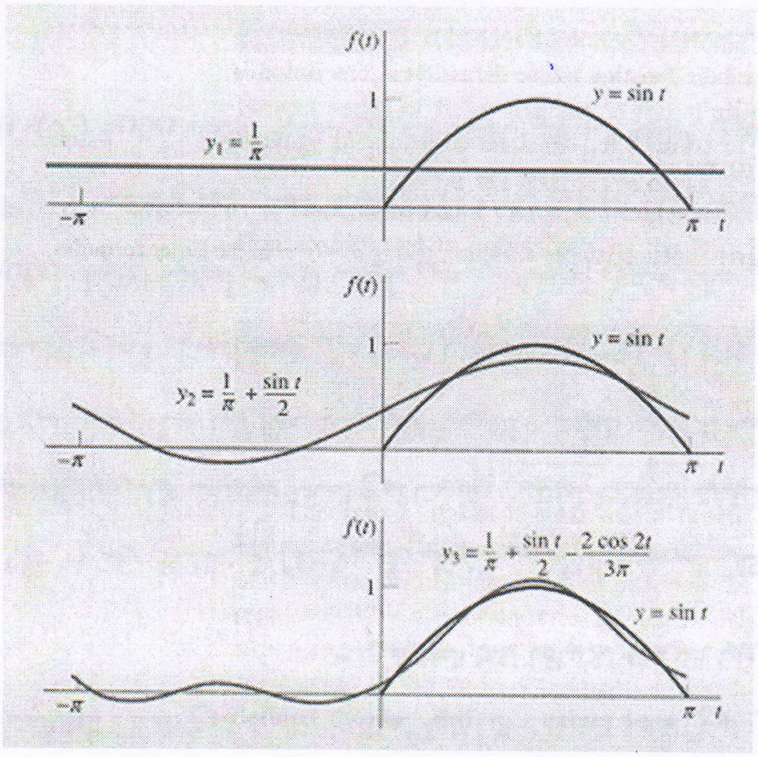
Since $\quad \sin t\sin nt = \frac{1}{2}[-\cos(1+n)\,t + \cos(1-n)\,t]$

$$b_n = \frac{1}{\pi}\left[\frac{1}{2}\left\{\frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1+n}\right\}\right]_0^{\pi} = 0 \qquad n \ne 1$$

For $n = 1$ $\quad b_1 = \frac{1}{\pi}\int_0^{\pi}\sin^2 t\,dt = \frac{1}{\pi}\int_0^{\pi}\frac{1-\cos 2t}{2}\,dt = \frac{1}{\pi}\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi} = \frac{1}{2}$

Hence, evaluating the coefficients for $= 0, 1, 2, 3, \cdots$, we have

$$f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi}\left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \cdots\right)$$



Theorem 1 If $f(t)$ is an even periodic function which satisfies the Dirichlet conditions, the coefficients in the Fourier series of $f(t)$ are given by the formulas

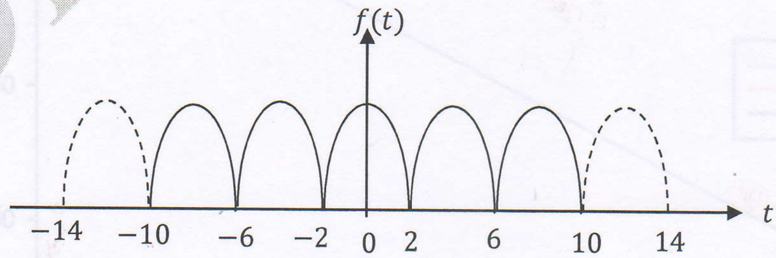
$$a_n = \frac{2}{p}\int_0^p f(t)\cos\frac{n\pi t}{p}\,dt \qquad b_n = 0 \qquad \text{where } 2p \text{ is the period of } f(t)$$

Example:- What is the Fourier expansion of the periodic function whose definition in one period is

$$f(t) = 4 - t^2 \qquad -2 \le t \le 2$$

Solution

Graph the given function, then from the graph the half-period of the given function is $p = 2$, taking



Since $f(t) = f(-t)$ the given function is even function then $b_n = 0$

$$a_n = \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}dt = \int_0^2 (4-t^2)\cos\frac{n\pi t}{2}dt$$

$$a_n = \int_0^2 4\cos\frac{n\pi t}{2}dt - \int_0^2 t^2\cos\frac{n\pi t}{2}d$$

$$a_n = \left[\frac{8}{n\pi}\sin\frac{n\pi t}{2}\right]_0^2 - \left[\frac{8t}{n^2\pi^2}\cos\frac{n\pi t}{2} + \left(\frac{2t^2}{n\pi} - \frac{16}{n^3\pi^3}\right)\sin\frac{n\pi t}{2}\right]_0^2$$

$$a_n = -\frac{16}{n^2\pi^2}\cos n\pi = \frac{16}{n^2\pi^2}(-1)^{n+1} \quad n \neq 0$$

For $n = 0$ $\quad a_o = \frac{2}{2}\int_0^2 (4-t^2)\,dt = \left[4t - \frac{t^3}{3}\right]_0^2 = \frac{16}{3}$

D		I
t^2	+	$\cos\frac{n\pi t}{2}$
$2t$	−	$\frac{2}{n\pi}\sin\frac{n\pi t}{2}$
2	+	$-\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi t}{2}$
0		$-\left(\frac{2}{n\pi}\right)^3\sin\frac{n\pi t}{2}$

Substituting these coefficients into the series, we obtain

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2}\left(\cos\frac{\pi t}{2} - \frac{1}{2^2}\cos\frac{\pi t}{2} + \frac{1}{3^2}\cos\frac{\pi t}{2} - \frac{1}{4^2}\cos\frac{\pi t}{2} + \cdots\right)$$

Theorem 2 If $f(t)$ is an odd periodic function which satisfies the Dirichlet conditions, the coefficients in the Fourier series of $f(t)$ are given by the formulas

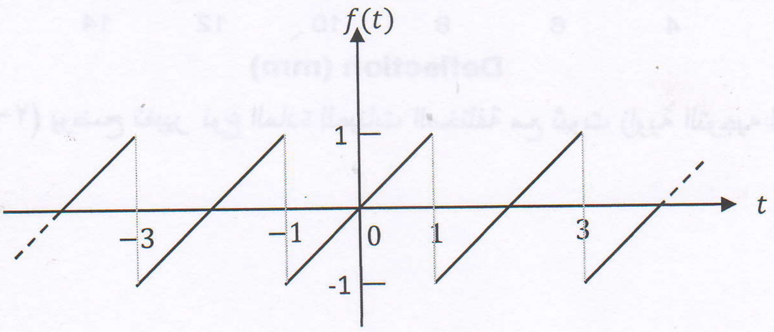
$$b_n = \frac{2}{p}\int_0^p f(t)\sin\frac{n\pi t}{p}dt \qquad a_n = 0 \qquad \text{where } 2p \text{ is the period of } f(t)$$

Example:- What is the Fourier expansion of the periodic function whose definition in one period is

$$f(t) = t \qquad -1 < t < 1$$

Solution

Graph the given function, then from the graph the half-period of the given function is $p = 1$



Since $f(t) = -f(-t)$ the given function is odd function then $a_n = 0$

$$b_n = \frac{2}{1}\int_0^1 f(t)\sin\frac{n\pi t}{1}dt = 2\int_0^1 t\cdot\sin n\pi t\ dt$$

$$b_n = 2\left[-t\cdot\frac{1}{n\pi}\cos n\pi t + \left(\frac{1}{n\pi}\right)^2\sin n\pi t\right]_0^1$$

$$b_n = 2\left[-\frac{1}{n\pi}\cos n\pi\right]$$

D		I
t	$+$	$\sin n\pi t$
1	$-$	$-\frac{1}{n\pi}\cos n\pi t$
0		$-\left(\frac{1}{n\pi}\right)^2\sin n\pi t$

Remember that $\cos n\pi = (-1)^n$

Hence $b_n = \frac{2(-1)^{n+1}}{n\pi}$

Substituting these coefficients into the series, we obtain

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi t}{n}$$

PROBLMES

Find the Fourier expansion of the periodic function whose definitions on one period is

1- $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 3 \\ -1 & 3 < t < 4 \end{cases}$ **Answers:** $a_n = 0$

$$b_n = \begin{cases} 2/\pi n & n = 1, 3, 5, \cdots \\ 4/\pi n & n = 2, 6, 10, \cdots \\ 0 & n = 4, 8, 12, \cdots \end{cases}$$

2- $f(t) = |t| \qquad -2 \le t \le 2$

3- $f(t) = \begin{cases} 0 & -2 \le t \le -1 \\ \cos\frac{\pi t}{2} & -1 \le t \le 1 \\ 0 & 1 \le t \le 2 \end{cases}$ **Answers:** $\frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi t}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos n\pi t}{4n^2-1}$

Half-Range Expansion

When $f(t)$ will be defined on an interval $0 \le t \le p$, and on this interval we want to represent $f(t)$ by a Fourier. Then, if we represents $f(t)$ an even periodic function which is called **half-range cosine series**

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi t}{p} \qquad b_n = 0$$

and if we represents $f(t)$ an odd periodic function which is called **half-range sine series,** then

$$f(t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi t}{p} \qquad a_n = 0$$