Solved Problems Part Three on

Adaptive Control

4^{th} Year Petroleum Systems

and Control Engineering

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Example

Given a plant that is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}$$

$$y = [1 \quad 0]\mathbf{x}$$

and has a performance index

$$J = \int_0^\infty \left[\mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \mathbf{u}^2 \right] dt$$

Let $$\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{R} = \text{scalar} = 1$$

Determine

(a) the Riccati matrix P

(b) the state feedback matrix K

Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{PA} + \mathbf{A}^{1}\mathbf{P} + \mathbf{Q} - \mathbf{PBR}^{-1}\mathbf{B}^{1}\mathbf{P} = \mathbf{0} \tag{1}$$

$$\mathbf{PA} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -p_{12} & p_{11} - 2p_{12} \\ -p_{22} & p_{21} - 2p_{22} \end{bmatrix} \tag{2}$$

$$\mathbf{A}^{T}\mathbf{P} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} -p_{21} & -p_{22} \\ p_{11} - 2p_{21} & p_{12} - 2p_{22} \end{bmatrix} \tag{3}$$

$$\begin{aligned} \mathbf{PBR}^{-1}\mathbf{B}^{T}\mathbf{P} &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 [0 \quad 1] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &= \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} [p_{21} \quad p_{22}] \\ &= \begin{bmatrix} p_{12}p_{21} & p_{12}p_{22} \\ p_{22}p_{21} & p_{22}^{2} \end{bmatrix} \end{aligned} \tag{4}$$

Combining equations (2, 3, and 4) gives:

$$\begin{bmatrix} -p_{12} & p_{11}-2p_{12} \\ -p_{22} & p_{21}-2p_{22} \end{bmatrix} + \begin{bmatrix} -p_{21} & -p_{22} \\ p_{11}-2p_{21} & p_{12}-2p_{22} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} p_{12}p_{21} & p_{12}p_{22} \\ p_{22}p_{21} & p_{22}^2 \end{bmatrix} = \mathbf{0} \qquad (5)$$

Since P is symmetric, p21 = p12. Equation (5) can be expressed as four simultaneous equations

$$-p_{12} - p_{12} + 2 - p_{12}^2 = 0 \qquad (6)$$

$$p_{11} - 2p_{12} - p_{22} - p_{12}p_{22} = 0 \qquad (7)$$

$$-p_{22} + p_{11} - 2p_{12} - p_{12}p_{22} = 0 \qquad (8)$$

$$p_{12} - 2p_{22} + p_{12} - 2p_{22} + 1 - p_{22}^2 = 0 \qquad (9)$$

Note that equations (7) and (8) are the same. From equation (6)

$$p_{12}^2 + 2p_{12} - 2 = 0$$

Solving using the formula: $P12 = P21 = \frac{-B \mp \sqrt{B^2 - 4AC}}{2A}$

$$p_{12} = p_{21} = 0.732 \quad \text{and} \quad -2.732$$

Using positive value

$$p_{12} = p_{21} = 0.732 \qquad 10$$

From equation (9)

$$2p_{12} - 4p_{22} + 1 - p_{22}^2 = 0$$

$$p_{22}^2 + 4p_{22} - 2.464 = 0$$

$$p_{22} = 0.542 \text{ and } -4.542$$

$$p_{22} = 0.542 \qquad 11$$

From equation (7)

$$p_{11} - (2 \times 0.732) - 0.542 - (0.732 \times 0.542) = 0$$

$$p_{11} = 2.403 \qquad (12)$$

From equations (10), (11) and (12) the Riccati matrix is

$$\mathbf{P} = \begin{bmatrix} 2.403 & 0.732 \\ 0.732 & 0.542 \end{bmatrix}$$

(b) Using the state feedback matrix equation

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} = 1[0 \quad 1]\begin{bmatrix} 2.403 & 0.732 \\ 0.732 & 0.542 \end{bmatrix}$$

$$\mathbf{K} = [0.732 \quad 0.542]$$

Deterministic SISO ARMA models

SISO ARMA model

$$A(q^{-1})\, y(k) = q^{-\mathrm{d}}\, B(q^{-1})\, [\, u(k) + d(k)]$$

Where all inputs and outputs are scalars:

- $u(k)$ control input
- $d(k)$ deterministic but unknown disturbance
- $y(k)$ output

Deterministic SISO ARMA models

$$A(q^{-1})\,y(k) = q^{-\mathsf{d}}\,B(q^{-1})\,[\,u(k) + d(k)]$$

Where polynomials:

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

are co-prime and d is the ***known*** pure time delay

Deterministic SISO ARMA models

We factor the zero polynomial as:

$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

where

$B^s(q^{-1})$ is anti-Schur

$B^u(q^{-1})$ has the zeros that we **do not want to cancel**

Control Objectives

1. **Pole Placement**: The poles of the closed-loop system must be placed at specific locations in the complex plane.

- **Closed-loop polynomial:**

$$\boxed{A_c(q^{-1}) = B^s(q^{-1})\, A_c'(q^{-1})}$$

Where:

- $B^s(q^{-1})$ cancelable plant zeros
- $A_c'(q^{-1})$ anti-Schur polynomial of the form

chosen by the designer

$$A_c'(q^{-1}) = \underset{\nwarrow}{1} + a_{c1}' q^{-1} + \cdots + a_{cn_c'}' q^{-n_c'}$$

Control Objectives

2. **Tracking**: The output sequence $y(k)$ must follow a ***reference*** sequence $y_d(k)$ which is known

In general, $y_d(k)$ can be generated by a reference model of the form

$$\underbrace{A_m}_{\text{anti-Schur polynomial}}(q^{-1})y_d(k) = q^{-\mathrm{d}}\, B_m(q^{-1})\, u_d(k)$$

The design of $A_m(q^{-1})$ and $B_m(q^{-1})$ is not a part of this control design technique and these polynomials do not enter into the analysis

Control Objectives

3. **Disturbance rejection**: The closed-loop system must reject a class of **persistent** disturbances $d(k)$

- **Disturbance model:**

$$\boxed{A_d(q^{-1})d(k) = 0}$$

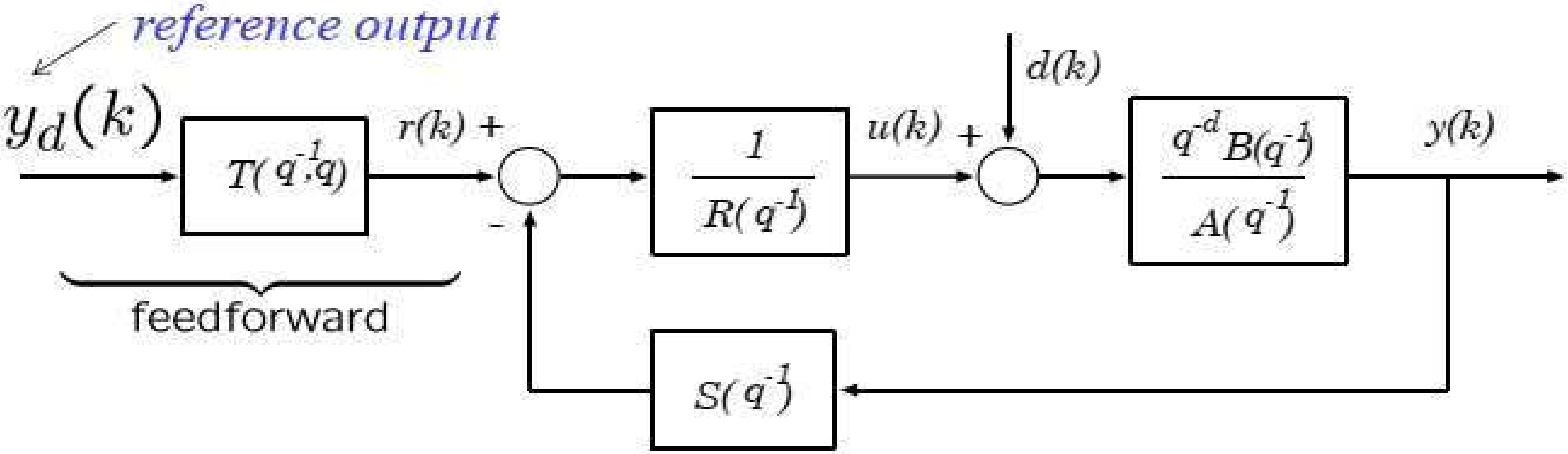
Where

- $A_d(q^{-1})$ is a ***known*** annihilating polynomial with zeros on the unit circle

- $A_d(q^{-1}),\ B(q^{-1})$ are co-prime

Control Law

- Feedback and feedforward actions:



$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$r(k) = T(q^{-1}, q)\, y_d(k)$$

Feedforward action (a-causal)

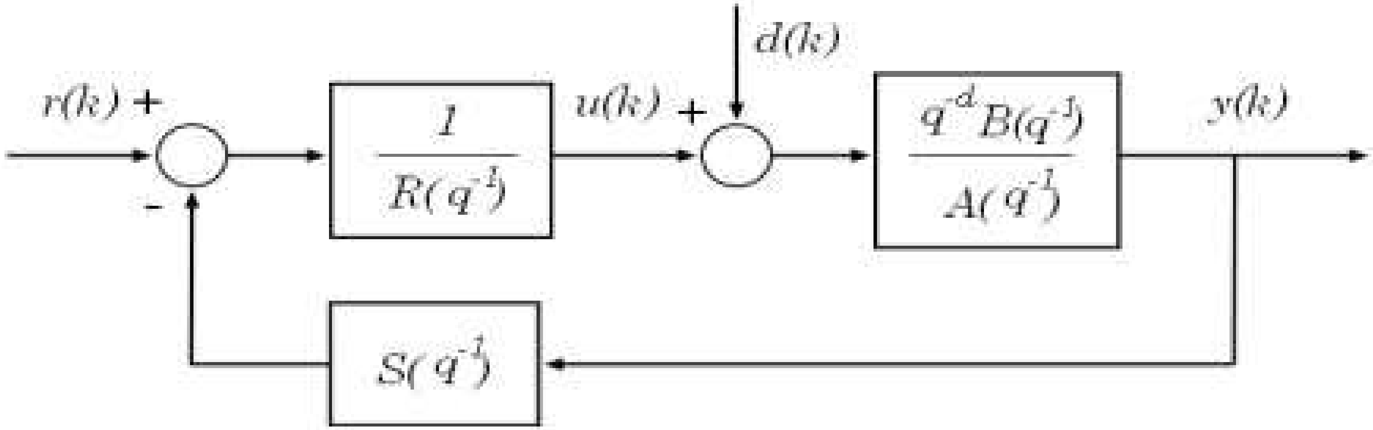
Feedback Controller

Diophantine equation: Obtain polynomials $\underline{R'(q^{-1})}, \underline{S(q^{-1})}$ that satisfy:

$$\boxed{\underbrace{A'_c(q^{-1})}_{\text{Close loop poles minus cancelled zeros}} = \underbrace{A_d(q^{-1})}_{\text{Disturbance annihilating polynomial}}\, \underbrace{A(q^{-1})}_{\text{Plant poles}}\, \underline{R'(q^{-1})} + q^{-\mathrm{d}}\underbrace{B^u(q^{-1})}_{\text{Unstable plant zeros}}\, \underline{S(q^{-1})}}$$

$$\boxed{\begin{aligned} R(q^{-1}) &= R'(q^{-1})\, A_d(q^{-1})\, B^s(q^{-1}) \\ A_c(q^{-1}) &= B^s(q^{-1})\, A'_c(q^{-1}) \end{aligned}}$$

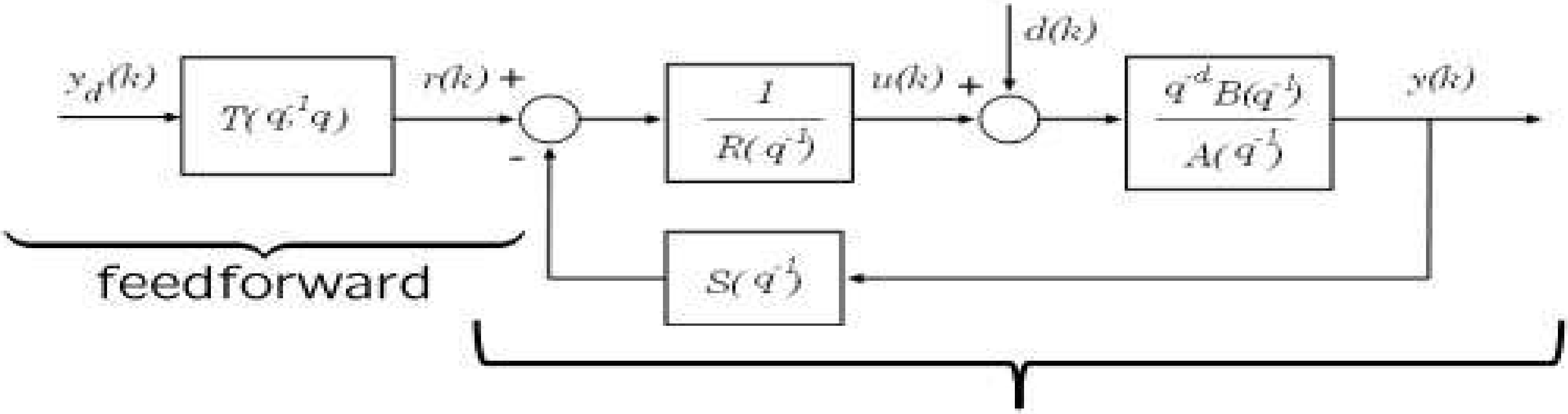
Feedback Controller



$$\boxed{y(k) = \frac{q^{-\mathrm{d}}B^{u}(q^{-1})}{A'_c(q^{-1})}\,r(k)}$$

$$\underbrace{+\ \frac{q^{-\mathrm{d}}B(q^{-1})\,R'(q^{-1})}{A_c(q^{-1})}\,A_d(q^{-1})d(k)}_{\to 0}$$

Zero-phase error feedforward



$$\frac{q^{-d}B^u(q^{-1})}{A'_c(q^{-1})}$$

$$\boxed{T(q^{-1},q) = A'_c(q^{-1})\, q^{+d}\frac{B^u(q)}{[B^u(1)]^2}}$$