Solved Problems Part Two on

Adaptive Control

4th Year Petroleum Systems and Control Engineering

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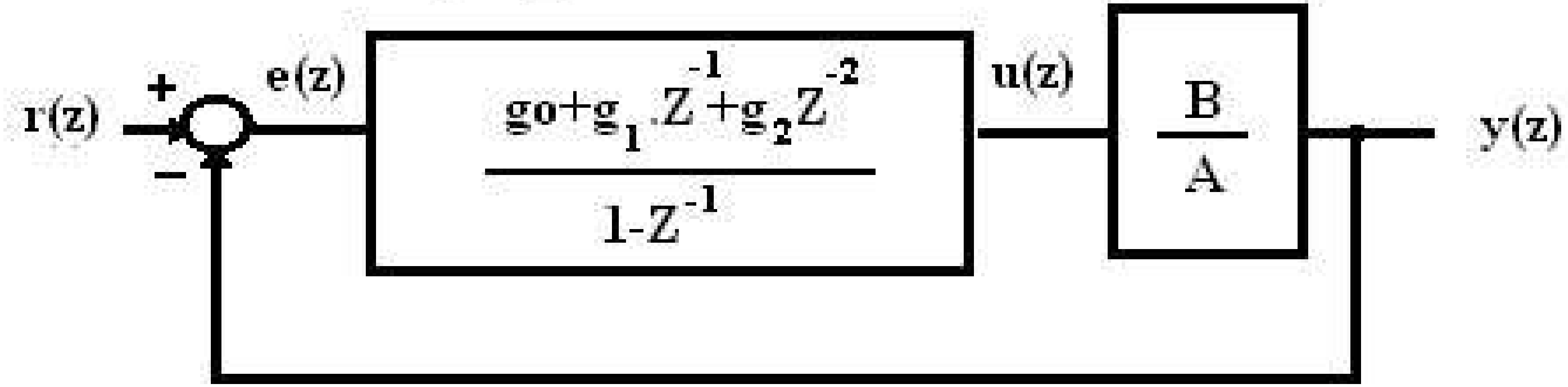
Three-term Controller design by pole-assignment:

Three-term or PID controller combines proportional action, integral action and derivative action and can put into the form:

$$u(z) = \frac{g_o + g_1.Z^{-1} + g_2.Z^{-2}}{1 - Z^{-1}} e(z) \qquad (1)$$



The coefficients g_o, g_1 and g_2 are related to the proportional derivative and integral gain setting, (assuming backward shift approximation) with T_s=1 second):

$K_p = -g_1 - 2.g_2$

$K_d = g_2$

$K_i = g_o + g_1 + g_2$

In order to synthesize exactly the PID controller coefficients, we can assume the system to be controlled has the following structure:

$$y(z) = \frac{b_1.Z^{-1}}{1 + a_1.Z^{-1} + a_2.Z^{-2}}.u(z) \qquad (2)$$

and we can assume that a1, a2 and b1 can be estimated by RLS method. The restriction on the system model form is to ensure that only one set of PID controller coefficients arises from the design. Combine equations (1) and (2), we get:

$$y(z)=\frac{b_{1}Z^{-1}.(g_o+g_1.Z^{-1}+g_2.Z^{-2}).r(z)}{(1-Z^{-1}).(1+a_1.Z^{-1}+a_2.Z^{-2})+(g_o+g_1.Z^{-1}+g_2.Z^{-2}).b_1.Z^{-1}} \qquad (3)$$

We can now select the coefficients g_o, g_1 and g_2 to give a desired closed loop performance. Suppose we have selected a desired closed loop the rise time and damped natural frequency and hence arrived at the corresponding desired closed loop T polynomial:

$$T=1+t_1.Z^{-1}+t_2.Z^{-2} \qquad (4)$$

Thus:

$$(1-Z^{-1}).(1+a_1.Z^{-1}+a_2.Z^{-2})+(g_o+g_1.Z^{-1}+g_2.Z^{-2}).b_1.Z^{-1}=1+t_1.Z^{-1}+t_2.Z^{-2}$$

(5)

By equating coefficients of like power of Z, we get:

$$g_o=\frac{t_1+(1-a_1)}{b_1},\ g_1=\frac{t_2+(a_1-a_2)}{b_1}\ \text{and}\ g_2=\frac{a_2}{b_1} \qquad (6)$$

The steady state matching of y(z) to r(z) for a constant reference signal is ensured by the factor (1 - Z^{-1}) in the denominator of equation (3). Specifically, for zero frequency (Z=1), so that independent of the system or controller parameters, we have y(z)=r(z) (for constant r(z)).

Example:

Drive the adjustment mechanism using MIT rule for MRAC with open loop Transfer Function of G(s) and reference model of Gm(s) = K0 G(s), where K0 is unknown parameter.

Solution:

$$e = y - y_m$$

$$y = G(s)\,\theta\,u_c$$

$$y_m = G(s)\,K_0\,u_c$$

$$u_c = \frac{y_m}{G(s)\,K_0}$$

$$e = G(s)\theta\,u_c - G(s)\,K_0\,u_c$$

The sensitivity derivative is:

$$\frac{\partial e}{\partial \theta} = G(s)\, u_c$$

$$\frac{\partial e}{\partial \theta} = G(s)\ \frac{y_m}{G(s)\, K_0} = \frac{y_m}{K_0}$$

apply MIT rule

$$\frac{d\theta}{dt} = -\gamma\, e\, \frac{\partial e}{\partial \theta}$$

$$\frac{d\theta}{dt} = -\gamma\ \frac{y_m\, e}{K_0}$$

Example:

Design a MRAC MIT rule using the following controller:

$$u(t) = q_0\, u_c(t) - w_0\, y(t)$$

Where $q_0 = \frac{b_m}{b}$ *and* $w_0 = \frac{a_m - a}{b}$, $a_m > a$

Given a system described by the model:

$$\frac{dy}{dt} = -a\, y + b\, u$$

Where u is the control variable and y is the measured output. Assume that it is desirable to obtain a closed loop system described by:

$$\frac{dy_m}{dt} = -a_m\, y_m + b_m\, u_c$$

With an error equation: $e = y - y_m$

Solution:

The transfer function of the system is:

$$s\,Y(s) = -a\,Y(s) + b\,U(s)$$

$$(s+a)\,Y(s) = b\,U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{b}{s+a} = G(s)$$

The transfer function of the reference model is:

$$\frac{Y_m\,(s)}{U_c\,(s)} = \frac{b_m}{s+a_m} = G_m(s)$$

The closed loop transfer function of the system is :

$$Y(s) = \frac{b\,q_0}{s+a+b\,w_0} U_c(s)$$

The error equation is:

$$e = y - y_m = \frac{b\,q_0\,U_c(s)}{s+a+b\,w_0} - \frac{b_m\,U_c(s)}{s+a_m}$$

The parameters of the controller are q_0 and w_0 applying MIT rule , we get:

$$\frac{\partial e}{\partial q_0} = .\frac{b\,U_c(s)}{s+a+b\,w_0}$$

And

$$\frac{\partial e}{\partial w_0} = \frac{-b^2\,q_0}{(s+a+b\,w_0)^2}\;U_c(s)$$

$$\frac{\partial e}{\partial w_0} = \frac{-b}{(s+a+b\,w_0)}\;Y(s)$$

The formula of $\frac{\partial e}{\partial q_0}$ and $\frac{\partial e}{\partial w_0}$ con not be used because the process parameters a and b are not Known. Approximation are therefore required in order to obtain realization parameters adjustment rule

And

$$\frac{dw_0}{dt} = +\gamma^{\backslash}\, b\ e\ \left(\frac{1}{s+a_m}\right)Y(s)$$

$$\frac{dw_0}{dt} = +\gamma\ e\ \left(\frac{1}{s+a_m}\right)Y(s)$$

Example:

Design Lyapunov based a MRAC using the following controller:

$$u(t) = q_0\, u_c(t) - w_0\, y(t)$$

Where $q_0 = \frac{b_m}{b}$ *and* $w_0 = \frac{a_m - a}{b}$, $a_m > a$

Given a system described by the model:

$$\frac{dy}{dt} = -a\, y + b\, u$$

Where u is the control variable and y is the measured output. Assume that it is desirable to obtain a closed loop system described by:

$$\frac{dy_m}{dt} = -a_m\, y_m + b_m\, u_c$$

With an error equation: $e = y - y_m$

Solution:

$$\frac{dy}{dt} = -a\,y + b\,u$$

$$\frac{dy_m}{dt} = -a_m\,y_m + b_m\,u_c$$

$$\frac{de}{dt} = \frac{dy}{dt} - \frac{dy_m}{dt}$$

$$\frac{de}{dt} = -a\,y + b\,u + a_m\,y_m - b_m\,u_c$$

$$\frac{de}{dt} = -a\,y + b\left(q_0\,u_c(t) - w_0\,y(t)\right) + a_m\,y_m - b_m\,u_c$$

$$\frac{de}{dt} = -a\,y + b\left(q_0\,u_c(t) - w_0\,y(t)\right) + a_m\,(y - e) - b_m\,u_c$$

$$\frac{de}{dt} = -a_m\,e + (a_m - a - b\,w_0)y + (b\,q_0 - b_m)\,u_c$$

The error goes to zero if the parameters are equal to the desired ones.

To derive the parameters q_0 and w_0, assume the Lyapunov function as:

$$V(e, q_0, w_0) = \frac{1}{2}\left(e^2 + \frac{1}{b\,\gamma}\,(b\,w_0 + a - a_m)^2 + \frac{1}{b\,\gamma}(b\,q_0 - b_m)^2\right)$$

This function is zero when (e) is zero and the controller parameters are equal to the optimal values. The derivative of V is:

$$\frac{dV}{dt} = e\,\frac{de}{dt} + \frac{1}{\gamma}\,(bw_0 + a - a_m)\,\frac{dw_0}{dt} + \frac{1}{\gamma}(b\,q_0 - b_m)\,\frac{dq_0}{dt}$$

Substitute $\frac{de}{dt}$ in $\frac{dV}{dt}$, $we\ get$:

$$\frac{dV}{dt} = -a_m\,e^2 + \frac{1}{\gamma}(bw_0 + a - a_m)\left(\frac{dw_0}{dt} - \gamma\,y\,e\right) + \frac{1}{\gamma}(b\,q_0 - b_m)\left(\frac{dq_0}{dt} + \gamma\,u_c\,e\right)$$

If parameters are updated as:

$$\frac{dq_0}{dt} = -\gamma\, u_c\, e$$

$$\frac{dw_0}{dt} = \gamma\, y\, e$$

We get:

$$\frac{dV}{dt} = -a_m\, e^2$$

Example: For a given plant dynamics design Linear Quadratic Optimal Control.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k), \;\; T = 1$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Where X1 and X2 are the system's states, u is the system's input, and K is an integer discrete time index.

Solution:

From the given plant dynamics the matrices A, B, and C can be determined:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \boldsymbol{T} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} \frac{\boldsymbol{T}^2}{\mathbf{2}} \\ \boldsymbol{T} \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}$$

Performance Index:

$$J = \frac{1}{2}\sum_{k=0}^{\infty}((y(k)-u_c)^T Q(y(k)-u_c)+u(k)^T Ru(k))$$

where J is a scalar performance index, Q is a positive semi-definite matrix, R is a positive definite matrix, and the command signal u_c is assumed to be constant.

The optimal control sequence that minimises J is:

$$u(k) = -K_f\, x(k) + K_r u_c$$

where the feedback gain matrix K_f is given by

$$K_f = (B^T S B + R)^{-1} B^T S A$$

where S is the solution to the algebraic Riccati equation

$$S = A^T [S - S B (B^T S B + R)^{-1} B^T S] A + Q$$

The feed-forward gain matrix K_r is given by:

$$K_r = (B^T S B + R)^{-1} B^T [I - (A - B K_f)^T]^{-1} C^T Q$$

The AutoRegressive with eXtra input (ARX) model

- given $y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b) + e(k)$
- introduce

$$B(q^{-1}) = b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b};\ A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n_a}$$

- we have

$$y(k) = \frac{B(q^{-1})}{A(q^{-1})} u(k) + \frac{1}{A(q^{-1})} e(k)$$

- i.e.

$$\underbrace{A(q^{-1})\, y(k)}_{\text{Autoregressive (AR)}} = \underbrace{B(q^{-1})\, u(k)}_{\text{Extra (X) input}} + e(k)$$

  - $e(k)$ directly enters the difference equation. The model is thus also named as the *equation error model*.

Linear regressor example for ARX

$$A\left(q^{-1}\right) y\left(k\right) = B\left(q^{-1}\right) u\left(k\right) + e\left(k\right)$$
$$B(q^{-1}) = b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}$$
$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n_a}$$

- adding $y\left(k\right)$ and substracting $A\left(q^{-1}\right) y\left(k\right)$ at each side gives

$$y\left(k\right) = \left[1 - A\left(q^{-1}\right)\right] y\left(k\right) + B\left(q^{-1}\right) u\left(k\right) + e\left(k\right)$$

- $[1 - A\left(q^{-1}\right)] y\left(k\right)$ depends on $y\left(k-1\right)$, $y\left(k-2\right)$, ... but not $y\left(k\right)$: $1 - A\left(q^{-1}\right) = -\left(a_1 q^{-1} + \cdots + a_n q^{-n_a}\right)$
- $\Rightarrow$a predictor form can thus be:

$$\hat{y}\left(k\right) = \left[1 - A\left(q^{-1}\right)\right] y\left(k\right) + B\left(q^{-1}\right) u\left(k\right)$$

The AutoRegressive Moving Average with eXtra input (ARMAX) model

- consider $y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b) + e(k) + c_1 e(k-1) + \cdots + c_{n_c} e(k-n_c)$
- i.e.

$$\underbrace{A\left(q^{-1}\right) y(k)}_{\text{autoregressive (AR)}} = \underbrace{B\left(q^{-1}\right) u(k)}_{\text{extra (X)}} + \underbrace{C\left(q^{-1}\right) e(k)}_{\text{moving average (MA)}}$$

$$\begin{aligned} B(q^{-1}) &= b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b} \\ A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n_a} \\ C(q^{-1}) &= 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c} \end{aligned}$$

we have

$$y(k) = \frac{B\left(q^{-1}\right)}{A\left(q^{-1}\right)} u(k) + \frac{C\left(q^{-1}\right)}{A\left(q^{-1}\right)} e(k)$$