Solved Problems Part One on

Adaptive Control

4th Year Petroleum Systems and Control Engineering

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Transfer Function to State Space Representations

Consider the dynamic of a system given in the following Transfer Function:

$$G(s) = \frac{y(s)}{u(s)} = \frac{6}{s^3 + 6\,s^2 + 11\,s + 6}$$

Then $y^{\backslash\backslash\backslash} + 6\,y^{\backslash\backslash} + 11\,y^{\backslash} + 6\,y = 6u$

Note: Number of states = the degree of the differential equation

We have 3 states:

$$x_1 = y \qquad , \qquad x_2 = y^{\backslash} \qquad , \qquad x_3 = y^{\backslash\backslash}$$

$taking\ the\ derivative\ with$ respect to time

$$x_1^{\backslash} = y^{\backslash} = x_2 \qquad , \qquad x_2^{\backslash} = y^{\backslash\backslash} = x_3$$

$$x_3^{\backslash} = y^{\backslash\backslash\backslash} = -6\,y - 11\,y^{\backslash} - 6\,y^{\backslash\backslash} + 6\,u$$

$$x_3^{\backslash} = y^{\backslash\backslash\backslash} = -6\,x_1 - 11\,x_2 - 6\,x_3 + 6\,u$$

$$\begin{bmatrix} x_1^{\backslash} \\ x_2^{\backslash} \\ x_3^{\backslash} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0\,u$$

$$x^{\backslash} = A\,x + B\,u \qquad (state\ equation)$$

$$y = C\,x + D\,u \qquad (outpot\ equation)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad , \quad D = 0$$

$$x^{\backslash} = \begin{bmatrix} x_1^{\backslash} \\ x_2^{\backslash} \\ x_3^{\backslash} \end{bmatrix} \quad , \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note: for (n) states with (m) inputs and (r) output:

A = n x n   in this case   3 x 3   (Dimension)

B = n x m   in this case   3 x 1   (Dimension)

C = r x n   in this case   1 x 3   (Dimension)

X = n x 1   in this case   3 x 1   (Dimension)

D = r x m   in this case   1 x 1   (Dimension)

State Space to Transfer Function Representations

Consider the dynamic of a system with State Space representation:

$$x^{\backslash} = A\,x + B\,u$$

$$y = C\,x + D\,u$$

Where y is the system output and u is the system input

the dynamic of a system can be presented by Transfer Function as:

$$G(s) = \frac{y(s)}{u(s)}$$

Taking Laplace Transform for the state space equations:

$$s\,X(s) - x(0) = A\,X(s) + B\,U(s) \qquad (1)$$

$$Y(s) = C\,X(s) + D\,U(s) \qquad (2)$$

From equation (1) assuming relaxed system (x(0) = 0)

$$s\,X(s) - A\,X(s) = B\,U(s)$$

$$(sI - A)X(s) = B\,U(s)$$

$$X(s) = (sI - A)^{-1}\,B\,U(s) \qquad (3)$$

Substitute (3) in (2):

$$Y(s) = C\,(sI - A)^{-1}\,B\,U(s) + D\,U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C\,(sI - A)^{-1}\,B + D$$

And can be written as:

$$G(s) = \frac{Q(s)}{(sI - A)}$$

$(sI - A)$ *is the characteristic polynomial of* $G(s)$. *In other words*

The eigenvalues of (A) = The poles of (G)

Question (1): Write the Transfer Function of a system given its state space representation as:

$$\begin{bmatrix} x_1^{\backslash} \\ x_2^{\backslash} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad , \quad D = 0$$

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2 + 3\,s + 2}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2 + 3\,s + 2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$G(s) = \frac{1}{s^2 + 3\,s + 2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$G(s) = \frac{1}{s^2 + 3\,s + 2}$$

Question (2):

Calculate the desired closed loop equation in discrete form for a second order system with $\omega_n = 10$, $\zeta = 0.7$ and the sampling time is 0.1 second.

Solution:

The standard form of a second order Transfer Function is:

$$\frac{U(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\,\zeta\,\omega_n\,s + w_n^2} = \frac{100}{s^2 + 2 * 0.7 * 10\,s + 100}$$

$$\frac{U(s)}{R(s)} = \frac{100}{s^2 + 14\,s + 100} = \frac{100}{(s + P1)(s + P2)}$$

$$P1\,, P2 = \frac{-14 \pm \sqrt{14^2 - 4 * 100}}{2} = -7 \pm j7.14$$

$$\frac{U(s)}{R(s)} = \frac{100}{(s+7+j7.14)(s+7-j7.14)}$$

$$Z = e^{T\,s} = e^{T\,P1,2} = e^{0.1*(-7\pm j7.14)} = e^{-0.7\pm j0.714}$$

Z1,2 =0.496*(0.755 $\pm$ j0.655) = 0.375 $\pm$ j0.325

$$T = (z - z_1)(z - z_2)$$

$$= (z\ - 0.375 + j0.325)(z - 0.375 - j0.325)$$

$$T = z^2 - 0.75\,z + 0.246$$

$$T = 1 - 0.75\,z^{-1} + 0.246\,z^{-2}$$

Question (3):

Derive the PID controller equation in z-domain using backward approximation.

Solution: The PID equation in S-plane is:

$$Gc(s) = Kp + \frac{Ki}{s} + \mathrm{Kds}$$

Using backward difference, the PID equation for $s = \frac{z-1}{Ts\,z}$ is:

$$Gc(z) = Kp + \frac{Ki\,Ts\,z}{z-1} + \frac{Kd\,(z-1)}{Ts\,z}$$

Assume Ts = 1 (special case)

$$Gc(z) = \frac{Kp\,(z-1)\,z + Ki\,z^2 + Kd\,(z-1)^2}{z\,(z-1)}$$

$$Gc(z) = \frac{Kp\ z^2 - Kp\ z + Ki\ z^2 + Kd z^2 - 2\ z\ Kd + Kd}{z^2 - z}$$

$$Gc(z) = \frac{(Kp + Kd\ z + Ki) z^2 - (Kp + 2\ Kd) z + Kd}{z^2 - z} \quad * \ \frac{z^{-2}}{z^{-2}}$$

$$Gc(z) = \frac{(Kp + Kd\ z + Ki) - (Kp + 2\ Kd) z^{-1} + Kd z^{-2}}{1 - z^{-2}}$$

$$Gc(z) = \frac{g_0 + g_1 z^{-1} + g_2 z^{-1}}{1 - z^{-1}}$$

Where

$$g_0 = (Kp + Ki + Kd) \qquad (1)$$

$$g_1 = -(Kp + 2\,Kd) \qquad (2)$$

$$g_2 = Kd \qquad (3)$$

Substitute (3) into (2), we get:

$$Kp = -g_1 - 2\,Kd$$

$$Kp = -g_1 - 2\,g_2 \qquad (4)$$

Substitute (3) and (4) in (1), we get:

$$g_0 = -g_1 - 2\,g_2 + g_2 + Ki$$

$$Ki = g_0 + g_1 + g_2$$

Pole assignment control:

It is a classical approach to controller specifications. The design of a feedback controller has two main aims:

1- the first is to modify in some way the dynamic response of a system.

2- The second is to reduce the sensitivity of the system output to Disturbances.

Suppose we have a system with a continuous time transfer function:

$$Y(s) = \frac{f}{1+\alpha.s} u(s) \qquad (1)$$

where α is the time constant of the system and f is the system gain.

The discrete-time domain transfer function is:

$$Y(z) = \frac{Z^{-1}.b}{1 - a.Z^{-1}} U(z) \qquad (2)$$

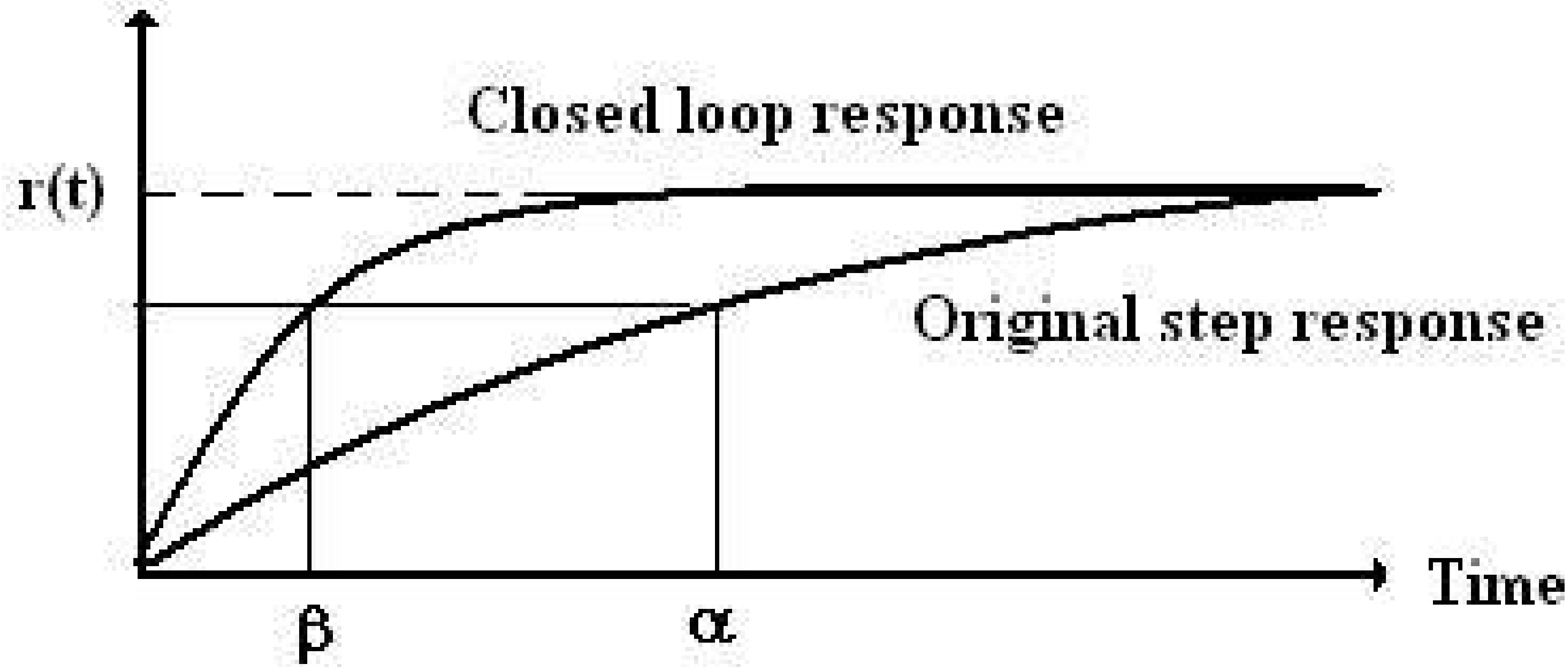
$$\text{where } a = e^{-T/\alpha} \text{ and } b = (1 - a).f \qquad (3)$$

and T is the sampling time.

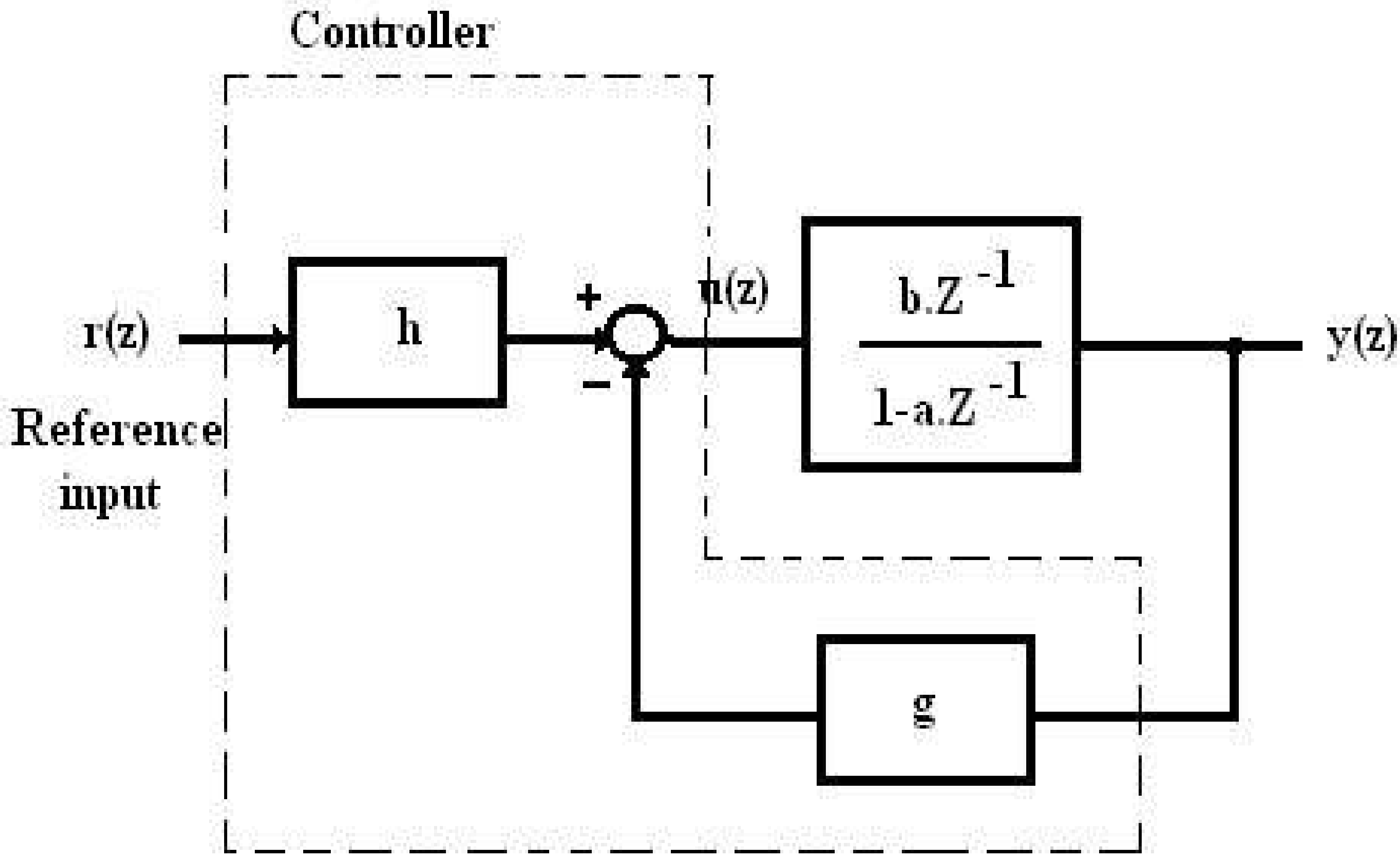
Common control objective would be to use a feedback controller to alter the speed of the system response. I.e. the original system might have a certain speed of response to a sudden change in u(z) associated with time constant α, but it is required to change this response rate to one associated with time constant β .

In terms of the system model, equation 2 changing the speed of response corresponds to later the value of a in the denominator in some way assigning it another value corresponding to faster speed of response.

Also, it is required to ensure that the output y(t) come into correspondence with reference signal r(t).



To do this, propose the feedback controller structure with the following block diagram:


Controller
r(z)
Reference input
h
+
−
u(z)
b.Z -1
1-a.Z -1
y(z)
g

Where: $$\mathbf{u(Z)=-g.y(Z)+h.r(Z)} \qquad (4)$$

By combining Eq. (2) and Eq. (4), this gives the closed loop transfer function:

$$\mathbf{y(z)=\frac{b.h.Z^{-1}}{1-(a-b.g).Z^{-1}}r(Z)} \qquad (5)$$

Notice that the closed loop speed of structure is determined by (a-b.g) instead of a. In other words, the open loop system pole Z=a has been assigned (placed) to a closed loop position Z=a-b.g. Hence, we can specify the location of the poles as a design parameter. Suppose we required that closed loop pole at Z=t1 , then:

$$\mathbf{t_1 = a - b.g} \qquad (6)$$

$$\mathbf{g=\frac{a-t_1}{b}} \quad \textbf{(Design equation)} \qquad (7)$$

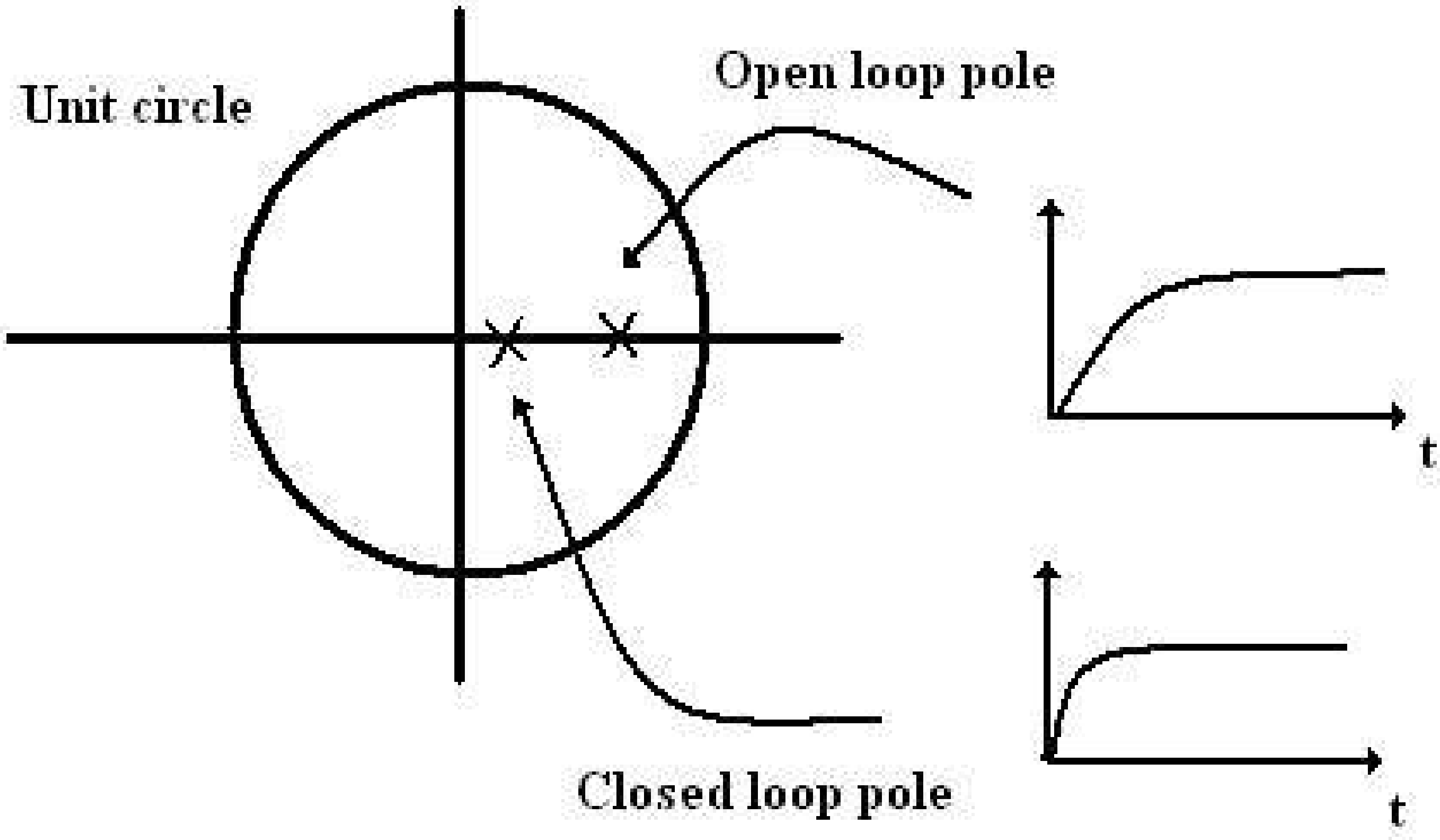
The remaining controller objective is to ensure that steady state (when r(z) is constant), the output y(z)=r(z).

This is done by selecting the controller parameter h such that the closed loop transfer function is equal to one at zero frequency. (Zero frequency means $s \to 0$ or $Z \to 1$). Using equation 5:

$$\frac{b.h}{1-(a-b.g)} = 1 \qquad (8)$$

$$\text{or } h = \frac{1-(a-b.g)}{b} \qquad (9)$$

Equation (9) is the design equation for selecting h so that y(z) tracks r(z) at steady state conditions. The key idea of pole assignment is to shift the open loop poles to some desired set of closed loop poles.


Unit circle
Open loop pole
Closed loop pole
t
t

Question (4):

Design a digital controller for the open loop discrete transfer function of $G_{OL}(z) = \frac{2}{Z-1}$, so that the digital closed loop transfer function has a zero steady state error with characteristic equation of $T = 1 - 0.5.Z^{-1}$.

Solution:

The open loop transfer function can be re-written as: $G_{OL}(z) = \frac{2Z^{-1}}{1-Z^{-1}}$

The closed loop transfer function is shown below:

$$y(z) = \frac{2.h.Z^{-1}}{1-(1-2.g).Z^{-1}} r(z)$$

By equating c\cs. equation with T, this gives:

1-2.g=0.5

2g=0.5

g=0.25

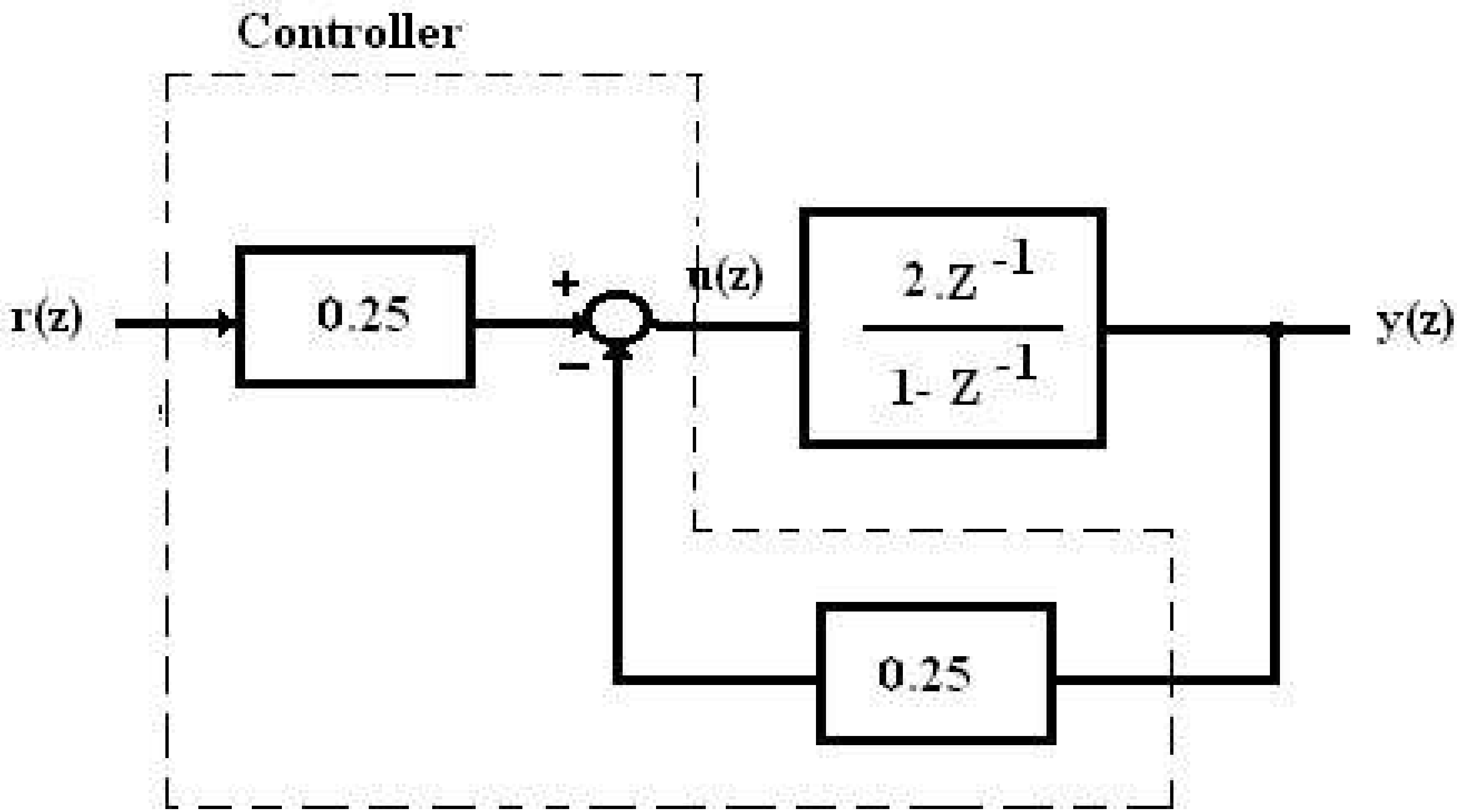
For zero steady state error,

$$\lim_{z \to 1} \frac{y(z)}{r(z)} = 1 = \frac{2.h}{1-0.5}$$

2.h=1-0.5

2.h=0.5

h=0.25



$$y(z) = \frac{0.5.Z^{-1}}{1-0.5.Z^{-1}} r(z)$$

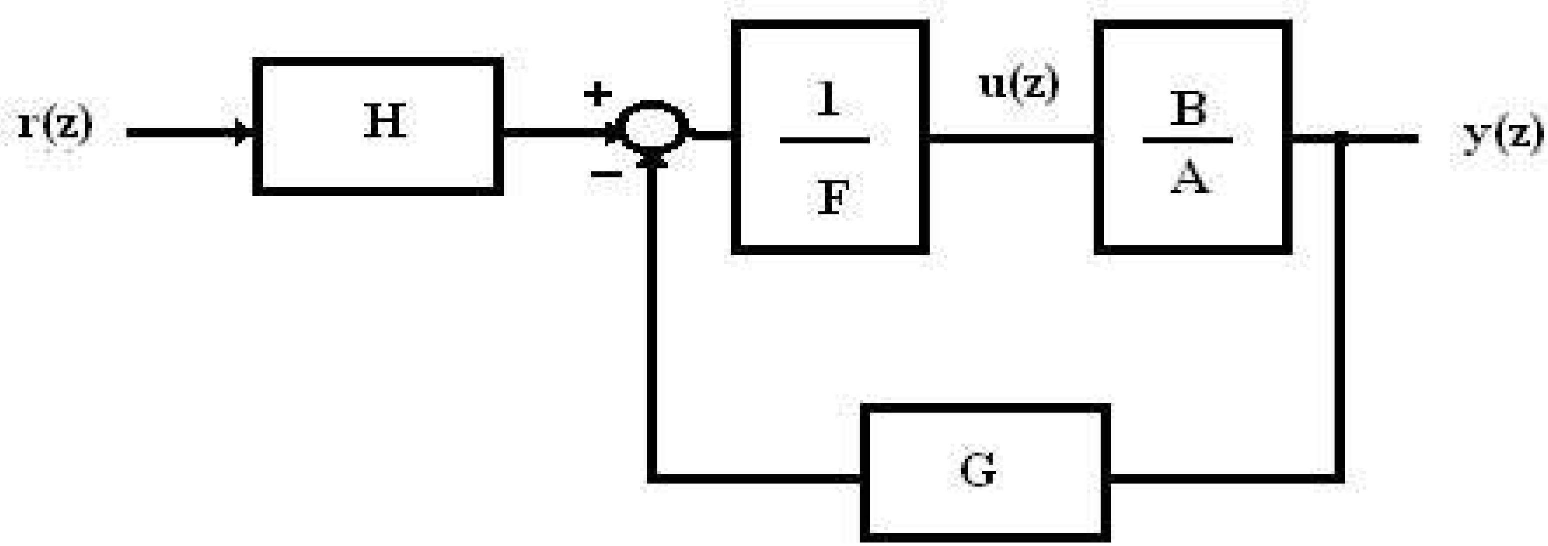
Question (5):

For the following model: $y(z)=\dfrac{Z^{-1}+2Z^{-2}}{1-1.7Z^{-1}+0.72Z^{-2}}u(z)$

Design a digital controller using pole placement that satisfies the following Closed loop characteristic equation: $T=1-0.904Z^{-1}$.

Solution:

The block diagram of the closed loop system is:



The closed loop Transfer Function is:

$$y(z) = \frac{B\ H}{F\ A + B\ G}\ r(z)$$

The design equation is:

$$F\ A + B\ G = T \qquad (1)$$

nt =1 , nb = 2 , na = 2 , nf = nb -1 = 2-1 = 1 , ng = na -1 = 2-1 = 1

nt <= na + nb – 1 ; nt <= 2+2-1 ; nt <= 3

The condition of nt is satisfied

$$F = 1 + f_1\ z^{-1}$$

$$G = g_0 + g_1\ z^{-1}$$

Substitute in equation (1):

$$(1 + f_1\, z^{-1})(1 - 1.7\, z^{-1} + 0.72\, z^{-2})$$

$$+ (z^{-1} + 2\, z^{-2})(g_0 + g_1\, z^{-1}) = 1 - 0.904\, z^{-1}$$

$$1 - 1.7\, z^{-1} + 0.72\, z^{-2} + f_1\, z^{-1} - 1.7\, f_1\, z^{-2} + 0.72\, f_1\, z^{-3}$$

$$+ g_0\, z^{-1} + g_1\, z^{-2} + 2\, g_0\, z^{-2} + 2\, g_1\, z^{-3} = 1 - 0.904\, z^{-1}$$

$$1 + (-1.7 + f_1 + g_0)z^{-1} + (0.72 - 1.7\, f_1 + g_1 + 2\, g_0)z^{-2}$$

$$+ (0.72\, f_1 + 2\, g_1)z^{-3} = 1 - 0.904\, z^{-1}$$

$$z^{-1} : -1.7 + f_1 + g_0 = -0.904 \qquad (2)$$

$$z^{-2} : 0.72 - 1.7\, f_1 + g_1 + 2\, g_0 = 0 \qquad (3)$$

$$z^{-3} : 0.72\, f_1 + 2\, g_1 = 0 \qquad (4)$$

From equation (2)

$$f_1 + g_0 = 0.796 \qquad (5)$$

From equation (3)

$$-1.7\, f_1 + 2\, g_0 + g_1 = -0.72 \qquad (6)$$

From equation (4)

$$0.72\, f_1 + 2\, g_1 = 0 \qquad (7)$$

Write equations (5-7) in matrix forms

$$\begin{bmatrix} 1 & 1 & 0 \\ -1.7 & 2 & 1 \\ 0.72 & 0 & 2 \end{bmatrix} \begin{bmatrix} f_1 \\ g_0 \\ g_1 \end{bmatrix} = \begin{bmatrix} 0.796 \\ -0.72 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & b_1 & 0 \\ a_1 & b_2 & b_1 \\ a\ \ 2 & 0 & b_2 \end{bmatrix} \begin{bmatrix} f_1 \\ g_0 \\ g_1 \end{bmatrix} = \begin{bmatrix} t_1 - a_1 \\ t_2 - a_2 \\ 0 \end{bmatrix}$$

A $\qquad\theta_c\qquad$ b

$$A\,\theta_c = b \quad then \quad \theta_c = A^{-1}\,b \qquad (8)$$

Calculate the A^{-1} using matrix theory

$$A^{-1} = \begin{bmatrix} 0.4926 & -0.2463 & 0.1232 \\ 0.5074 & 0.2463 & -0.1232 \\ -0.1773 & 0.0887 & 0.4557 \end{bmatrix}$$

$$\theta_c = \begin{bmatrix} f_1 \\ g_0 \\ g_1 \end{bmatrix} = \begin{bmatrix} 0.5695 \\ 0.2265 \\ -0.205 \end{bmatrix}$$

$$F = 1 + 0.5695\, z^{-1}$$

$$G = 0.2265 - 0.205\, z^{-1}$$

The closed loop Transfer Function is:

$$\frac{y(z)}{r(z)} = \frac{(z^{-1} + 2\, z^{-2})\, H}{1 - 0.904\, z^{-1}}$$

Given that steady state error is zero; using final value theorem

$$\lim_{z\to 1}\left(\frac{(z^{-1}+2\,z^{-2})\,H}{1-0.904\,z^{-1}}\right) = 1 \quad then \quad \frac{(1+2)\,h}{1-0.904}$$

$$h = \frac{1-0.904}{3} = 0.032 \qquad H\ becomes\ h\ for\ constant\ value$$

Another solution: Assume the requirement to cancel the zero (z+2)

let $H = \frac{z}{z+2} = \frac{1}{1+2\,z^{-1}}$

$$\frac{y(z)}{r(z)} = \frac{(z^{-1}+2\,z^{-2})}{(1-0.904\,z^{-1})(1+2\,z^{-1})} = \frac{z^{-1}\,(1+2\,z^{-1})}{(1-0.904\,z^{-1})(1+2\,z^{-1})}$$

$$\frac{y(z)}{r(z)} = \frac{z^{-1}}{1-0.904\,z^{-1}}$$