2.3.2 Buckingham's method (or Π-Theorem)

The Buckingham's Π-theorem is based on the following steps:
Step 1. Identify the relevant variables
Step 2. Write down dimensions.
Step 3. Establish the number of independent dimensions and non-dimensional groups.
Number of relevant variables: $n = 6$
Number of independent dimensions: $m = 3$ (M, L and T)
Number of non-dimensional groups (Πs): $n - m = 3$
Step 4. Choose m (= 3) dimensionally-independent scaling variables
Step 5. Create the Πs by non-dimensionalising the remaining variables

Guidelines for choosing *repeating parameters* in step 4 of the method of repeating variables

• The dependent variable should not be selected as repeating variable.
• The dependent variables should be chosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.
Example:
Geometric properties: length, diameter, height
Flow properties: velocity, acceleration
Fluid properties: viscosity, density
• The repeating variable should not form a dimensionless group
• The repeating variables should have the same number of fundamental dimensions.
• No two repeating variables should have the same fundamental dimensions.

Example (2.5)

By dimensional analysis, obtain an expression for the drag force (F) on a partially submerged body moving with a relative velocity (u) in a fluid; the other variables being the linear dimension (L), surface roughness (ε), fluid density (ρ), and gravitational acceleration (g).

Solution:

$$F = f(u, L, \varepsilon, \rho, g)$$

Drag force (F) N $\equiv [MLT^{-2}]$
Relative velocity (u) m/s $\equiv [LT^{-1}]$
Linear dimension (L) m $\equiv [L]$
Surface roughness (ε) m $\equiv [L]$
Density (ρ) $\equiv [ML^{-3}]$
Acceleration of gravity (g) m/s^2 $\equiv [L\ T^{-2}]$
$n = 6$, $m = 3$, $\Rightarrow \Pi = n - m = 6 - 3 = 3$

No. of repeating variables = m = 3
The selected repeating variables is (u, L, ρ)

$\Pi_1 = u^{a1} L^{b1} \rho^{c1} F$ --------------(1)
$\Pi_2 = u^{a2} L^{b2} \rho^{c2} \varepsilon$ --------------(2)
$\Pi_3 = u^{a3} L^{b3} \rho^{c3} g$ --------------(3)

For Π1 equation (1)
$[M^0 L^0 T^0] = [L\ T^{-1}]^{a1} [L]^{b1} [ML^{-3}]^{c1} [MLT^{-2}]$

Now applied dimensional homogeneity
For M $0 = c1 + 1$ $\Rightarrow c1 = -1$
For T $0 = -a1 - 2$ $\Rightarrow a1 = -2$
For L $0 = a1 + b1 - 3c1 + 1$ $\Rightarrow b1 = -2$
$\Pi_1 = u^{-2} L^{-2} \rho^{-1} F$

$$\Pi_1 = \frac{F}{\rho u^2 L^2}$$

For Π_2 equation (2)
$[M^0 L^0 T^0] = [L\ T^{-1}]^{a2} [L]^{b2} [ML^{-3}]^{c2} [L]$

For M $0 = c2$ $\Rightarrow c2 = 0$
For T $0 = -a2$ $\Rightarrow a2 = 0$
For L $0 = a2 + b2 - 3c2 + 1$ $\Rightarrow b2 = -1$

$\Pi_2 = L^{-1} \varepsilon$

$$\Pi_2 = \frac{\varepsilon}{L}$$

For Π_3 equation (3)

$[M^0 L^0 T^0] = [L T^{-1}]^{a3} [L]^{b3}[ML^{-3}]^{c3}[L T^{-2}]$

For M $0 = c3$ $\Rightarrow c3 = 0$

For T $0 = -a3 - 2$ $\Rightarrow a3 = -2$

For L $0 = a3 + b3 - 3c3 + 1$ $\Rightarrow b3 = 1$

$$\Pi_3 = u^{-2} L g$$

$$\Pi_3 = \frac{L g}{u^2}$$

$\Pi_1 = f_1 (\Pi_2, \Pi_3,)$

$$F = \rho u^2 L^2 f(\frac{\varepsilon}{L}, \frac{L g}{u^2})$$

Example (2.6)

Show that the discharge of a centrifugal pump is given by

$$Q = ND^3 f(\frac{gH}{N^2 D^2} \frac{\mu}{ND^2 \rho})$$

where (N) is the speed of the pump in r.p.m., (D) the diameter of impeller, (g) gravitational acceleration, (H) manometric head, (μ), (ρ) are the dynamic viscosity and the density of the fluid.

Solution:

$$Q = f (N, D, g, H, \mu, \rho)$$

Discharge (Q) m^3/s $\equiv [L^3T^{-1}]$

Pump speed (N) r.p.m. $\equiv [T^{-1}]$

Diameter of impeller (D) m $\equiv [L]$

Acceleration of gravity (g) m/s^2 $\equiv [L T^{-2}]$

Head of manometer (H) m $\equiv [L]$

Viscosity (μ) kg/m.s $\equiv [ML^{-1} T^{-1}]$

Density (ρ) kg/m^3 $\equiv [ML^{-3}]$

$n = 7$, $m = 3$, $\Rightarrow \Pi = n - m = 7 - 3 = 4$

No. of repeating variables = m = 3

The selected repeating variables is (N, D, ρ)

$\Pi_1 = N^{a1}\, D^{b1}\, \rho^{c1}\, Q$ --------------(1)

$\Pi_2 = N^{a2}\, D^{b2}\, \rho^{c2}\, g$ --------------(2)

$\Pi_3 = N^{a3}\, D^{b3}\, \rho^{c3}\, H$ --------------(3)

$\Pi_4 = N^{a4}\, D^{b4}\, \rho^{c4}\, \mu$ --------------(4)

For Π_1 equation (1)

$[M^0\, L^0\, T^0] = [T^{-1}]^{a1}\, [L]^{b1} [ML^{-3}]^{c1} [L^3\, T^{-1}]$

For M $0 = c1$ $\Rightarrow c1 = 0$

For T $0 = -a1 - 1$ $\Rightarrow a1 = -1$

For L $0 = b1 - 3c1 + 3$ $\Rightarrow b1 = -3$

$\Pi_1 = N^{-1}\, D^{-3}\, Q$

$$\Pi_1 = \frac{Q}{ND^3}$$

For Π_2 equation (2)

$[M^0\, L^0\, T^0] = [T^{-1}]^{a2}\, [L]^{b2} [ML^{-3}]^{c2} [LT^{-2}]$

For M $0 = c2$ $\Rightarrow c2 = 0$

For T $0 = -a2 - 2$ $\Rightarrow a2 = -2$

For L $0 = b2 - 3c2 + 1$ $\Rightarrow b2 = -1$

$\Pi_2 = N^{-2}\, D^{-1}\, g$

$$\Pi_2 = \frac{g}{DN^2}$$

For Π_3 equation (3)

$[M^0\, L^0\, T^0] = [T^{-1}]^{a3}\, [L]^{b3} [ML^{-3}]^{c3} [L]$

For M $0 = c3$ $\Rightarrow c3 = 0$

For T $0 = -a3$ $\Rightarrow a3 = 0$

For L $0 = b3 - 3c3 + 1$ $\Rightarrow b3 = -1$

$\Pi_3 = D^{-1}\, H$

$$\Pi_3 = \frac{H}{D}$$

For Π_3 equation (4)

$[M^0\ L^0\ T^0] = [T^{-1}]^{a4}\ [L]^{b4}[ML^{-3}]^{c4}[ML^{-1}T^{-1}]$

For M $0 = c4 + 1$ $\Rightarrow c4 = -1$

For T $0 = -a4 - 1$ $\Rightarrow a4 = -1$

For L $0 = b4 - 3c4 - 1$ $\Rightarrow b4 = -2$

$\Pi_4 = N^{-1}\ D^{-2}\ \rho^{-1}\ \mu$

$$\Pi_4 = \frac{\mu}{ND^2\,\rho}$$

$\Pi_1 = f_1(\Pi_2, \Pi_3, \Pi_4)$

$$Q = ND^3\, f\left(\frac{\mu}{ND^2\,\rho}, \frac{gH}{D^2\,N^2}\right)$$

2.5 Dimensions of some important variables

Item	Property	Symbol	SI Units	M.L.T.
1-	**Velocity**	**u**	**m/s**	LT^{-1}
2-	**Angular velocity**	**ω**	**Rad/s, Deg/s**	T^{-1}
3-	**Rotational velocity**	**N**	**Rev/s**	T^{-1}
4-	**Acceleration**	**a, g**	m/s^2	LT^{-2}
5-	**Angular acceleration**	**α**	s^{-2}	T^{-2}
6-	**Volumetric flow rate**	**Q**	m^3/s	L^3T^{-1}
7-	**Discharge**	**Q**	m^3/s	L^3T^{-1}
8-	**Mass flow rate**	***m***	**kg/s**	MT^{-1}
9-	**Mass (flux) velocity**	**G**	$kg/m^2.s$	$ML^{-2}T^{-1}$
10-	**Density**	**ρ**	kg/m^3	ML^{-3}
11-	**Specific volume**	**υ**	m^3/kg	L^3M
12-	**Specific weight**	**sp.wt**	N/m^3	$ML^{-2}T^{-2}$

13-	**Specific gravity**	**sp.gr**	**[-]**	**[-]**
14-	**Dynamic viscosity**	**μ**	**kg/m.s, Pa.s**	$ML^{-1}T^{-1}$
15-	**Kinematic viscosity**	**ν**	m^2/s	L^2T^{-1}
16-	**Force**	**F**	**N**	MLT^{-2}
17-	**Pressure**	**P**	**N/m2≡Pa**	$ML^{-1}T^{-2}$
18-	**Pressure gradient**	**ΔP/L**	**Pa/m**	$ML^{-2}T^{-2}$
19-	**Shear stress**	**τ**	N/m^2	$ML^{-1}T^{-2}$
20-	**Shear rate**	**γ**	**s-1**	T^{-1}
21-	**Momentum**	**M**	**kg.m/s**	MLT^{-1}
22-	**Work**	**W**	**N.m ≡J**	ML^2T^{-2}
23-	**Moment**	**M**	**N.m ≡J**	ML^2T^{-2}
24-	**Torque**	**Γ**	**N.m ≡J**	ML^2T^{-2}
25-	**Energy**	**E**	**J**	ML^2T^{-2}
26-	**Power**	**P**	**J/s ≡W**	ML^2T^{-3}
27-	**Surface tension**	**σ**	**N/m**	MT^{-2}
28-	**Efficiency**	**η**	**[-]**	**[-]**
29-	**Head**	**h**	**m**	**L**
30-	**Modulus of elasticity**	**ε, K**	**Pa**	$ML^{-1}T^{-2}$