Dimensional Analysis

A dimension is a measure of a physical quantity without numerical values, while a unit a way to assign a number to the dimension. For example length is a dimension but cm is a unit.

2.1 Fundamentals Dimensions

Dimensions can be classified as either fundamental or derived.

Fundamental dimensions include:

Mass	(kg)	dimension (M)
Length	(m)	dimension (L)
Time	(s)	dimension (T)
Temperature	(^{o}c)	dimension (θ)

Derived dimensions can be expressed in terms of fundamental dimensions, for example,

Area	(m^2)	dimension (L^2)
Volume	(m^3)	dimension (L^3)
Force F	(N = kg m/ s^2)	dimension (ML^1T^{-2})
Energy E	(J = kg m^2/ s^2)	dimension (ML^2T^{-2})
Power HP	(W =kg m^2 /s^3)	dimension (ML^2T^{-3})
Viscosity μ	(kg/m s)	dimension ($ML^{-1}T^{-1}$)
Shear stress τ	(N/m^2)	dimension (ML^1T^{-2})

2-2 Dimensional Homogeneity

If an equation have the same dimensions that both right and left hand sides of the equation .This is also known as the law of *dimensional homogeneity*.

Let consider the common equation of volumetric flow rate.

$Q = A\ u$

$L^3T^{-1} = L^2\ .LT^{-1} = L^3T^{-1}$

Example (2.1)

Check the dimensional homogeneity of the following equations

1- $\mathbf{u} = \sqrt{\dfrac{\mathbf{2g(\rho_{m}-\rho)\Delta z}}{\boldsymbol{\rho}}}$

2- $Q = \dfrac{8}{15} cd\ tan\dfrac{\emptyset}{2}\sqrt{2g}\ Z^{5/2}$

1) $\mathbf{u} = \sqrt{\frac{\mathbf{2g(\rho_{m-}\rho)\Delta z}}{\boldsymbol{\rho}}}$

L.H.S. u ≡ [LT^{-1}]

R.H.S. u ≡ $\left[\frac{L\,T^{-2}(ML^{-3})L}{ML-^{3}}\right]^{1/2}$ = [LT^{-1}]

Since the dimensions on both sides of the equation are same, therefore the equation is *dimensionally homogenous*.

2) $Q = \frac{8}{15} cd\ tan\frac{\emptyset}{2}\sqrt{2g}\ Z^{5/2}$

L.H.S. Q ≡ [L^3T^{-1}]

R.H.S. $(LT^{-2})^{1/2}\ (L)^{5/2}$ ≡ [L^3T^{-1}]

This equation is dimensionally homogenous.

Example (2-2)

If P is pressure and u is velocity and ρ is density what are the dimension in (MLT system) for a) P/ρ b) pρu c) $p/\rho u^2$

Solution:

$a)\qquad \frac{P}{\rho} = \frac{ML^{-1}T^{-2}}{ML^{-3}} = L^2T^{-2}$

$B)\qquad ML^{-1}T^{-2}.ML^{-3}.LT^{-2} = M^2L^{-3}T^{-3}$

$c)\qquad \frac{P}{\rho u^2} = \frac{ML^{-1}T^{-2}}{ML^{-3}(LT^{-1})^2} = M^0L^0T^0 \quad dimensionless$

2-3 Methods of Dimensional Analysis

Many methods of dimensional analysis are available; two of these methods are given here, which are:

1- Rayleigh's method (or Power series)

2- Buckingham's method (or Π-Theorem)

2.3.1 Rayleigh's method (or Power series)

The Rayleigh's method is based on the following steps:-

1- First of all, write the functional relationship with the given data.

2- Write the equation in terms of a constant with exponents i.e. powers a, b, c...

3- With the help of the principle of dimensional homogeneity, find out the values of a, b, c … by obtaining simultaneous equation and simplify it.

4-Now substitute the values of these exponents in the main equation, and simplify it.

Example (2.3)

Prove that the resistance (F) of a sphere of diameter (d) moving at a constant speed (u) through a fluid density (ρ) and dynamic viscosity (μ) may be expressed as:

Solution:

Resistance (F) N $\equiv [MLT^{-2}]$
Diameter (d) m $\equiv [L]$
Speed (u) m/s $\equiv [LT^{-1}]$
Density (ρ) kg/m^3 $\equiv [ML^{-3}]$
Viscosity (μ) kg/m.s $\equiv [ML^{-1} T^{-1}]$

$$F = f(d, u, \rho, \mu)$$
$$F = k(d^a, u^b, \rho^c, \mu^d)$$
$$[MLT^{-2}] = [L]^a [LT^{-1}]^b [ML^{-3}]^c [ML^{-1}T^{-1}]^d$$

For M $\quad 1 = c + d \quad \Rightarrow \quad c = 1 - d$ ----------------(1)

For L $\quad 1 = a + b - 3c - d$ ----------------(2)

For T $\quad -2 = -b - d \quad \Rightarrow b = 2 - d$ ----------------(3)

By substituting equations (1) and (3) in equation (2) give
$a = 1 - b + 3c + d = 1 - (2 - d) + 3(1 - d) + d = 2 - d$

$F = k(d^{2-d}, u^{2-d}, \rho^{1-d}, \mu^{d})$

$$F = k(d^2u^2\rho)\left[\frac{\mu}{\rho\, u\, d}\right]^d$$

Instead of this form where **<u>k and d</u>** unknown quantities, the above equation is written in functional form

$$F = f(d^2u^2\rho)\left[\frac{\mu}{\rho\, u\, d}\right]$$

<u>Example (2-4)</u>

The pressure difference (AP) between two ends of a pipe in which a fluid is flowing is a function of the pipe diameter *d,* the pipe length L, the fluid velocity *u*, the fluid density *p,* and the fluid viscosity μ.

Solution:

$$\triangle p = f(L, d, \rho, u, , \mu)$$

$$\triangle p = k(L^a, d^b, \rho^c, u^d, \mu^e)$$

$$[ML^{-1}\, T^{-2}] = [L]^a\,[L]^b[ML^{-3}]^c\,[LT^{-1}]^d[ML^{-1}T^{-1}]^e$$

For M $\quad 1 = c + e \quad \Rightarrow c = 1 - e$ --------(1)
For L $\quad -1 = a + b - 3c + d - e$ --------(2)
For T $\quad -2 = -d - e \quad \Rightarrow d = 2 - e$ --------(3)

By substituting equations (1) and (3) in equation (2) give

$-1 = a + b - 3(1-e) + (2-e) - e$

$0 = a + b + e$

$b = -a - e$

$$\triangle p = (L^a, d^{-a-e}, \rho^{1-e}, u^{2-e}\, \mu^e)$$

$$\Delta p = k(\frac{L}{d})^a(\frac{\mu}{\rho u d})^e \rho u^2$$

Instead of this form where **k ,a and e** unknown quantities, the above equation is written in functional form

$$\Delta p = f(\frac{L}{d})(\frac{\mu}{\rho u d}) \rho u^2$$