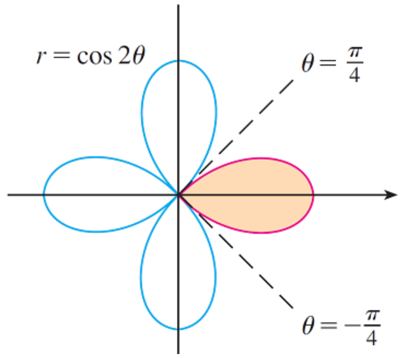
EXAMPL E: Find the area enclosed by one loop of the four - leaved rose r= co s 2θ.



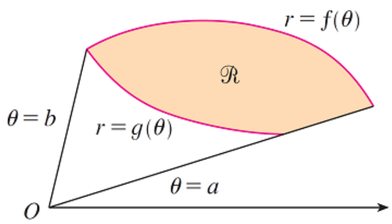
Solution: Notice that the region enclosed by the right loop is swept out by a ray that rotates from θ=−π/4 to θ =π /4 . Therefore, Formula 4 gives

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta = \int_0^{\pi/4} \cos^2 2\theta d\theta \\ &= \int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta) d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

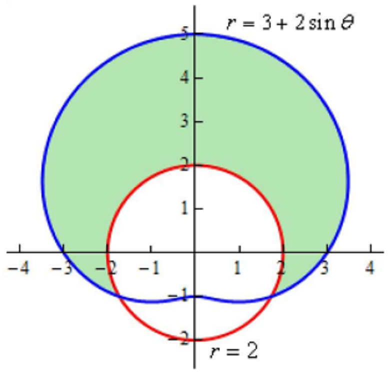
Let R be the region bounded by curves with polar equations r = f(θ), r = g(θ), θ = a, and θ = b, where

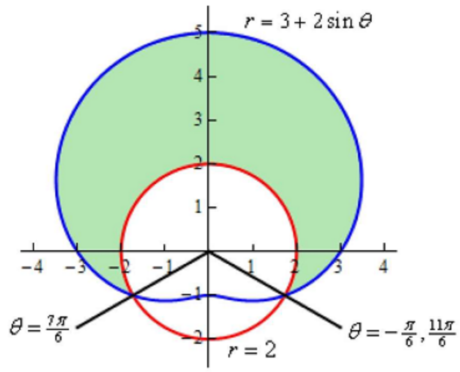
f(θ) ≥ g(θ) ≥ 0 and 0 < b−a ≤ 2π .Then the area A of R is

$$\boxed{A = \int_a^b \frac{1}{2}\left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta}$$



EXAMPL E: Find the area that lies insider= 3 + 2 sinθ and out side r= 2.





Solution: We first find a and b:

$$3 + 2\sin\theta = 2 \quad \Longrightarrow \quad \sin\theta = -\frac{1}{2} \quad \Longrightarrow \quad \theta = \frac{7\pi}{6},\ -\frac{\pi}{6}\left(\frac{11\pi}{6}\right)$$

Therefore the area is

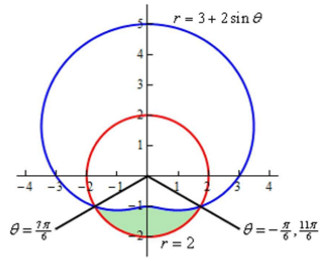
$$A = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[(3+2\sin\theta)^2 - 2^2\right] d\theta = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(5 + 12\sin\theta + 4\sin^2\theta)d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(7 + 12\sin\theta - 2\cos(2\theta))d\theta = \frac{1}{2}\Big[7\theta - 12\cos\theta - \sin(2\theta)\Big]_{-\pi/6}^{7\pi/6}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} \approx 24.187$$

EXAMPLE: Find the area of the region out side r= 3 + 2 sinθ and inside r= 2.

Solution: We have

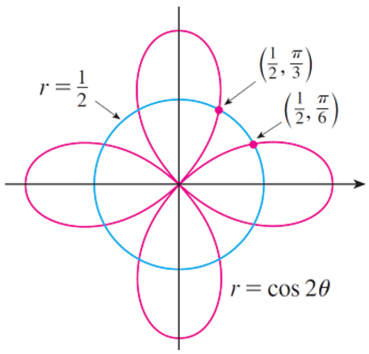


$$A = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}\left[2^2 - (3+2\sin\theta)^2\right] d\theta$$

$$= \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(-5 - 12\sin\theta - 4\sin^2\theta)d\theta$$

$$= \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(-7 - 12\sin\theta + 2\cos(2\theta))d\theta = \frac{1}{2}\Big[-7\theta + 12\cos\theta + \sin(2\theta)\Big]_{7\pi/6}^{11\pi/6} = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} \approx 2.196$$

EXAMPLE: Find all points of intersection of the curves r= co s 2θ and r =1/2



Solution: If we solve the equations r= cos2θ and r=1/2 , we get cos 2θ=1/2 and, therefore, 2θ=π /3, 5π /3, 7π /3, 11π /3 Thus the values of θ between 0 and 2π that satisfy both equations are θ=π /6, 5π /6, 7π /6, 1 1π /6 We have found four points of intersection:

$$\left(\frac{1}{2}, \pi/6\right), \ \left(\frac{1}{2}, 5\pi/6\right), \ \left(\frac{1}{2}, 7\pi/6\right), \text{ and } \left(\frac{1}{2}, 11\pi/6\right)$$

However, you can see from the above figure that the curves have four other points of intersection — namely,

$$\left(\frac{1}{2}, \pi/3\right), \ \left(\frac{1}{2}, 2\pi/3\right), \ \left(\frac{1}{2}, 4\pi/3\right), \text{ and } \left(\frac{1}{2}, 5\pi/3\right)$$

These can be found using symmetry or by noticing that another equation of the circle is r = −1/2 and then solving the equations r= co s 2θ and r= −1/2 .

Arc Length

To find the length of a polar curve r =f(θ), a ≤ θ ≤ b, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

So, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+\left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

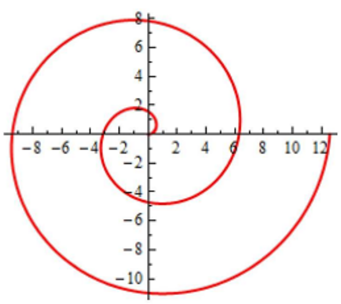
Assuming that f' is continuous, we can use one of the formulas to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore, the length of a curve with polar equation r= f(θ), a ≤ θ ≤ b, is

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

EXAMPL E: Find the length of the curve r=θ , 0 ≤ θ ≤ 1.



Solution: We have

$$L = \int_0^1 \sqrt{\theta^2+1}\,d\theta = \begin{bmatrix} \theta = \tan x \quad \Longrightarrow \quad \sqrt{\theta^2+1} = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = |\sec x| = \sec x \\ d\theta = d\tan x \\ d\theta = \sec^2 x dx \end{bmatrix}$$

$$= \int_0^{\pi/4} \sec^3 x dx = \left[\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|)\right]_0^{\pi/4} = \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2}))$$