$$\sum_{i=0}^{n-1}\left(\sum_{j=0}^{m-1} f(x_j, y_i)\Delta x\right)\Delta y.$$

In the sum in parentheses, only the value of $x_j$ is changing; $y_i$ is temporarily constant. As m goes to infinity, this sum has the right form to turn into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1} f(x_j, y_i)\Delta x = \int_a^b f(x, y_i)\,dx.$$

So after we take the limit as m goes to infinity, the sum is

$$\sum_{i=0}^{n-1}\left(\int_a^b f(x, y_i)\,dx\right)\Delta y.$$

Of course, for different values of $y_i$ this integral has different values; in other words, it is really a function applied to $y_i$:

$$G(y) = \int_a^b f(x, y)\,dx.$$

If we substitute back into the sum we get

$$\sum_{i=0}^{n-1} G(y_i)\Delta y.$$

This sum has a nice interpretation. The value $G(y_i)$ is the area of a cross section of the region under the surface f(x, y), namely, when y = $y_i$. The quantity $G(y_i)$_y can be interpreted as the volume of a solid with face area $G(y_i)$ and thickness _y. Think of the surface f(x, y) as the top of a loaf of sliced bread. Each slice has a cross-sectional area and a thickness; $G(y_i)$_y corresponds to the volume of a single slice of bread. Adding these up approximates the total volume of the loaf. (This is very similar to the technique

we used to compute volumes in section 9.3, except that there we need the cross-sections to be in some way “the same”.) Figure 15.1.3 shows this “sliced loaf” approximation using the same surface as shown in figure 15.1.2. Nicely enough, this sum looks just like the sort of sum that turns into an integral, namely,

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1} G(y_i)\Delta y &= \int_c^d G(y)\,dy\\ &= \int_c^d \int_a^b f(x,y)\,dx\,dy.\end{aligned}$$

Let’s be clear about what this means: we first will compute the inner integral, temporarily treating y as a constant. We will do this by finding an anti-derivative with respect to x, then substituting x = a and x = b and subtracting, as usual. The result will be an expression with no x variable but some occurrences of y. Then the outer integral will be an ordinary one-variable problem, with y as the variable.

**EXAMPLE 15.1.1** Figure 15.1.2 shows the function $\sin(xy)+6/5$ on $[0.5,3.5]\times[0.5,2.5]$. The volume under this surface is

$$\int_{0.5}^{2.5}\int_{0.5}^{3.5} \sin(xy)+\frac{6}{5}\,dx\,dy.$$

The inner integral is

$$\int_{0.5}^{3.5} \sin(xy)+\frac{6}{5}\,dx = \frac{-\cos(xy)}{y}+\frac{6x}{5}\Big|_{0.5}^{3.5} = \frac{-\cos(3.5y)}{y}+\frac{\cos(0.5y)}{y}+\frac{18}{5}.$$

Unfortunately, this gives a function for which we can’t find a simple anti-derivative. To complete the problem we could use Sage or similar software to approximate the integral.

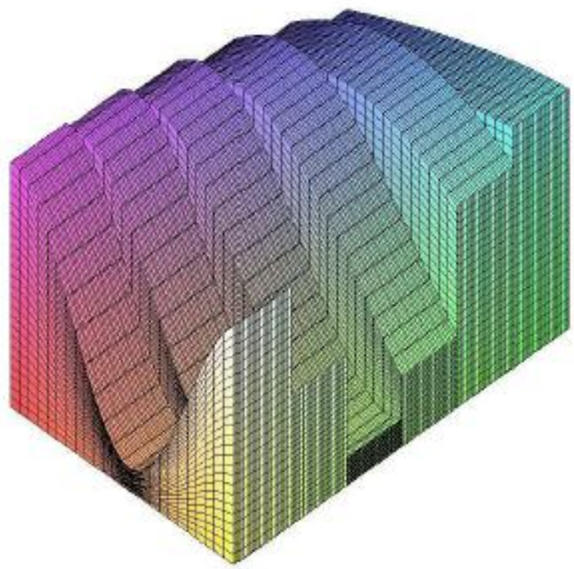

**Figure 15.1.3** Approximating the volume under a surface with slices.

Doing this gives a volume of approximately 8.84, so the average height is approximately 8.84/6 ≈ 1.47.

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j, y_i)\Delta x\Delta y = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} f(x_j, y_i)\Delta y\Delta x.$$

Now if we repeat the development above, the inner sum turns into an integral:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_j, y_i)\Delta y = \int_c^d f(x_j, y)\,dy,$$

and then the outer sum turns into an integral:

$$\lim_{m\to\infty}\sum_{j=0}^{m-1}\left(\int_c^d f(x_j, y)\,dy\right)\Delta x = \int_a^b\int_c^d f(x, y)\,dy\,dx.$$

In other words, we can compute the integrals in either order, first with respect to x then

y, or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven't really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called Fubini's Theorem.

**EXAMPLE 15.1.2** We compute $\displaystyle\iint_R 1+(x-1)^2+4y^2\,dA$, where $R=[0,3]\times[0,2]$, in two ways.

First,

$$\begin{aligned}
\int_0^3\int_0^2 1+(x-1)^2+4y^2\,dy\,dx &= \int_0^3 y+(x-1)^2y+\frac{4}{3}y^3\Big|_0^2\,dx\\
&= \int_0^3 2+2(x-1)^2+\frac{32}{3}\,dx\\
&= 2x+\frac{2}{3}(x-1)^3+\frac{32}{3}x\Big|_0^3\\
&= 6+\frac{2}{3}\cdot 8+\frac{32}{3}\cdot 3-(0-1\cdot\frac{2}{3}+0)\\
&= 44.
\end{aligned}$$

In the other order:

$$\begin{aligned}
\int_0^2\int_0^3 1+(x-1)^2+4y^2\,dx\,dy &= \int_0^2 x+\frac{(x-1)^3}{3}+4y^2x\Big|_0^3\,dy\\
&= \int_0^2 3+\frac{8}{3}+12y^2+\frac{1}{3}\,dy\\
&= 3y+\frac{8}{3}y+4y^3+\frac{1}{3}y\Big|_0^2\\
&= 6+\frac{16}{3}+32+\frac{2}{3}\\
&= 44.
\end{aligned}$$