Partial Derivatives

A real-valued function $w = f(x_1, x_2, \ldots, x_n)$

where w is the dependent variable of f &

f is a function of the n independent variable x_1 to x_n

also x_j called input variables & w called the ~~out~~ output variable, to understand this state, let us say that the volume of cylinder is a function of it's radius & height, hence.

$V = f(r, h)$ with a formula $V = \pi r^2 h$ where

r, h would be the independent variables & V is the dependent variable of the function

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- Partial derivatives of a function of two variables

we define the partial derivatives of f with respect to x at the point (x_0, y_0) as

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

as we hold the y-value constant at y_0 so y isn't a variable

- The partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

as we hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0,y)$ with respect to y at y_0

- other notation of partial derivative with respect to x

$$\frac{\partial f}{\partial x}(x_0,y_0) \text{ or } f_x(x_0,y_0),\ \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)},\ \&\ f_x,\ \frac{\partial f}{\partial x},\ z_x \text{ or } \frac{\partial z}{\partial x}$$

- other notation of partial derivative with respect to y

$$\frac{\partial f}{\partial y}(x_0,y_0) \text{ or } f_y(x_0,y_0),\ \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)},\ \&\ f_y,\ \frac{\partial f}{\partial y},\ z_y \text{ or } \frac{\partial z}{\partial y}$$

* ايجاد $\frac{\partial f}{\partial x}$ يتم عن طريق اعتبار y كأنها ثابتة ونشتق الدالة بالنسبة لـ x كما تعلمنا في الاشتقاق الاعتيادي (ordinary derivative)

* لإيجاد $\frac{\partial f}{\partial y}$ نعتبر x كأنها ثابتة ونشتق الدالة بالنسبة لـ y كما تعلمنا في الاشتقاق الاعتيادي والامثلة التالية توضح ذلك.

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Ex] Find the values of $\frac{df}{dx}$ & $\frac{df}{dy}$ at the point $(4,-5)$

For $f(x,y) = x^2 + 3xy + y - 1$

Sol] $$\frac{df}{dx} = \frac{d}{dx}(x^2+3xy+y-1) = 2x+3y+0-0$$

$$= 2x+3y$$

$$\left.\frac{df}{dx}\right|_{(4,-5)} = 2(4)+3(-5) = 8-15 = -7.$$

$$\frac{df}{dy} = \frac{d}{dy}(x^2+3xy+y-1) = 0+3x+1-0 = 3x+1$$

$$\left.\frac{df}{dy}\right|_{(4,-5)} = 3(4)+1 = 13$$

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Ex] Find $\frac{df}{dy}$ as a function if $f(x,y) = y \cdot \sin xy$

Sol] $$\frac{df}{dy} = \frac{d}{dy}(y \sin xy) = y \cdot \frac{d}{dy}\sin xy + (\sin xy)\frac{d}{dy}y$$

$$= y(\cos xy)\,x + (\sin xy)\cdot 1 = xy\cos xy + \sin xy$$

Ex] Find f_x & f_y as a functions if $f(x,y) = \frac{2y}{y+\cos x}$

$$f_x = \frac{d}{dx}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{d}{dx}(2y) - 2y\frac{d}{dx}(y+\cos x)}{(y+\cos x)^2}$$

$$f_x = \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{d}{dy}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{d}{dy}(2y) - 2y\frac{d}{dy}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(2) - 2y(1)}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}$$

* يمكن تطبيق الاشتقاق الضمني (implicit diffrentiation) بنفس الطريقة التي يطبق بها في الاشتقاق الاعتيادي كما مبين في المثال التالي

Ex] Find $\frac{dz}{dx}$ if the eq. $yz - \ln z = x + y$

sol] defines z as a function of the two independent variables x & y, hence we differentiate both sides of the eq. with respect to x, holding y constant & treating z as a differentiable function of x

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$$\frac{d}{dx}(y \cdot z) - \frac{d}{dx}\ln z = \frac{dx}{dx} + \frac{dy}{dx}$$

$$y\frac{dz}{dx} - \frac{1}{z}\frac{dz}{dx} = 1 + 0$$

$$\frac{dz}{dx}\left(y - \frac{1}{z}\right) = 1 \Longrightarrow \frac{dz}{dx} = \frac{1}{\left(y - \frac{1}{z}\right)} = \frac{1}{\frac{yz-1}{z}}$$

$$\therefore \frac{dz}{dx} = \frac{z}{yz-1}$$

Functions of more than two Variables

إجراء الاشتقاق الجزئي للدوال التي يكون فيها أكثر من متغيرين مستقلين يكون بنفس الآلية السابقة حيث نشتق الدالة بالنسبة للمتغير المطلوب ونعتبر بقية المتغيرات كأنها ثوابت.

Ex] if x, y & z are independent variables & $f(x,y,z) = x \cdot \sin(y+3z)$, find $\frac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x \sin(y+3z)) = x \frac{\partial}{\partial z}(\sin(y+3z)) + \sin(y+3z)\frac{\partial x}{\partial z}$$

$$= x\cos(y+3z) \cdot 3 + \sin(y+3z)(0) = 3x\cos(y+3z).$$

Ex] Find f_x, f_y & f_z for $f(x,y,z) = (x^2+y^2+z^2)^{\frac{-1}{2}}$

$$f_x = \frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}} * (2x+0+0) = -x(x^2+y^2+z^2)^{\frac{-3}{2}}.$$

$$f_y = \frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}} \cdot (0+2y+0) = -y \cdot (x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$f_z = \frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}} \cdot (0+0+2z) = -z(x^2+y^2+z^2)^{-3/2}.$$

Ex] Find f_x, f_y, f_z for $f(x,y,z) = \sin^{-1}(xyz)$.

sol] $f_x = \frac{\partial}{\partial x}(\sin^{-1}(xyz)) = \frac{1}{\sqrt{1-x^2y^2z^2}} \cdot \frac{\partial}{\partial x}(xyz)$

$$= \frac{yz}{\sqrt{1-x^2y^2z^2}}$$

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$$f_y = \frac{\partial}{\partial y}\left(\sin^{-1}(xyz)\right) = \frac{1}{\sqrt{1-x^2y^2z^2}}\frac{\partial}{\partial y}(xyz)$$

$$= \frac{xz}{\sqrt{1-x^2y^2z^2}}$$

$$f_z = \frac{\partial}{\partial z}\left(\sin^{-1}(xyz)\right) = \frac{1}{\sqrt{1-x^2y^2z^2}}\frac{\partial}{\partial z}(xyz)$$

$$= \frac{xy}{\sqrt{1-x^2y^2z^2}}$$

Ex} if you have a function $w = (xy)^z$ find w_x, w_y & w_z.

Sol} $w_x = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left((xy)^z\right) = y^z \cdot z x^{z-1}$

$$w_y = \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left((xy)^z\right) = x^z \cdot z y^{z-1}$$

$$w_z = \frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left((xy)^z\right) = (xy)^z \cdot \ln(xy)$$

Second-Order Partial Derivatives.

when we differentiate a function $f(x,y)$ twi.. , we produce it's second order derivatives which usually denoted by

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx} , $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

$\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx} & $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}

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$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \rightarrow$$ (Mixed partial derivatives)

$\frac{\partial^2 f}{\partial x \partial y}$ → differentiate first with respect to y, then with respect to x.

$\frac{\partial^2 f}{\partial y \partial x}$ → differentiate first with respect to x, then with respect to y.

حفظ

The mixed derivative theorem → $f_{xy}(a,b) = f_{yx}(a,b)$.

Ex) Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \frac{e^y}{y^2+1}$

Sol)

في البداية نشتق بالنسبة لـ y (نعتبر x ثابتة) ومن ثم نشتق بالنسبة لـ x (نعتبر y ثابتة)

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(xy + \frac{e^y}{y^2+1}\right) = y \longrightarrow \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$$

$$\boxed{\frac{\partial^2 w}{\partial y \partial x} = 1} \longleftarrow = \frac{\partial}{\partial y}(y) = 1$$

ولنتأكد من صحة النظرية أعلاه سنشتق $\frac{\partial^2 w}{\partial x \partial y}$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(xy + \frac{e^y}{y^2+1}\right) = \left(x + \frac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2}\right.$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(x + \frac{(y^2+1)e^y - 2ye^y}{(y^2+1)^2}\right)$$

$$\therefore \frac{\partial^2 w}{\partial x \partial y} = 1 \Longrightarrow \boxed{\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}}$$

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Ex} If $f(x,y) = x\cos y + y e^{x}$, find the second-order derivatives $\frac{d^2 f}{dx^2}$, $\frac{d^2 f}{dy\,dx}$, $\frac{d^2 f}{dy^2}$ & $\frac{d^2 f}{dx\,dy}$.

Sol}

$$\frac{df}{dx} = \frac{d}{dx}(x\cos y + y e^{x}) = \cos y + y e^{x}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx^2}\right) = \frac{d}{dx}(\cos y + y e^{x}) = y e^{x}.$$

$$\frac{d^2 f}{dy\,dx} = \frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d}{dy}(\cos y + y e^{x}) = -\sin y + e^{x}$$

$$\frac{df}{dy} = \frac{d}{dy}(x\cos y + y e^{x}) = -x\sin y + e^{x}$$

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$$\frac{d^2 f}{dy^2} = \frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d}{dy}(-x\sin y + e^{x}) = -x\cos y$$

$$\frac{d^2 f}{dx\,dy} = \frac{d}{dx}\left(\frac{df}{dy}\right) = \frac{d}{dx}(-x\sin y + e^{x}) = -\sin y + e^{x}$$

$$\frac{d^2 f}{dx\,dy} = \frac{d^2 f}{dy\,dx} = -\sin y + e^{x}$$ لاحظ هنا

Partial Derivatives of still Higher Order

We can differentiate a function as long as the derivatives involved exist, we get third & fourth-order derivatives denoted by symbols like $\frac{d^3 f}{dx\,dy^2} = f_{yyx}$, $\frac{d^4 f}{dx^2 dy^2} = f_{yyxx}$

Ex] Find f_{yxyz} if $f(x,y,z)=1-2xy^2z+x^2y$

Sol] $f_{yxyz}=\frac{d}{dz}\left(\frac{d}{dy}\left(\frac{d}{dx}\left(\frac{df}{dy}\right)\right)\right)$

هنا نبدأ بالاشتقاق بالنسبة لـ y ثم بالنسبة لـ x ثم بالنسبة لـ y واخيراً بالنسبة لـ z

$f(x,y,z)=1-2xy^2z+x^2y$.

$f_y=\frac{df}{dy}=\frac{d}{dy}(1-2xy^2z+x^2y)=-4xyz+x^2$.

$f_{yx}=\frac{d}{dx}\left(\frac{df}{dy}\right)=\frac{d}{dx}(-4xyz+x^2)=-4yz+2x$

$f_{yxy}=\frac{d}{dy}(f_{yx})=\frac{d}{dy}(-4yz+2x)=-4z$.

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$f_{yxyz}=\frac{d}{dz}(f_{yxy})=\frac{d}{dz}(-4z)=-4$.

Ex] Find all the second order partial derivatives of the function $w=x^2\cdot\tan(xy)$

Sol] $\frac{dw}{dx}=x^2\cdot\sec^2(x\cdot y)\cdot y+\tan(xy)\cdot 2x=x^2y\sec^2(xy)+2x\tan(xy)$

$\frac{dw}{dy}=x^2\sec^2(x\cdot y)\cdot x=x^3\sec^2(xy)$.

$\frac{d^2w}{dx^2}=\frac{d}{dx}\left(\frac{dw}{dx}\right)=\frac{d}{dx}\left(x^2y\sec^2(xy)+2x\tan(xy)\right)$.

$=x^2y\cdot 2\sec(xy)\cdot\tan(xy)\cdot\sec(xy)\cdot y+\sec^2(xy)\cdot 2xy+2x\sec^2(xy)\cdot y+$ $\tan(xy)\cdot 2$

$$\therefore \frac{d^2w}{dx^2} = 2x^2y^2 \sec^2(xy)\cdot\tan(xy) + 4xy\sec^2(xy) + 2\tan(xy)$$

$$\frac{d^2w}{dy^2} = \frac{d}{dy}\left(\frac{dw}{dy}\right) = \frac{d}{dy}\left(x^3\sec^2(xy)\right)$$

$$= x^3 \cdot 2\sec(xy)\cdot\tan(xy)\sec(xy)\cdot x + \sec^2(xy)\cdot(0)$$

$$= 2x^4\sec^2(xy)\tan(xy).$$

$$\frac{d^2w}{dy\,dx} = \frac{d}{dy}\left(\frac{dw}{dx}\right) = \frac{d}{dy}\left(x^2y\sec^2(xy) + 2x\tan(xy)\right)$$

$$= x^2\left(y\cdot 2\sec(xy)\cdot\tan(xy)\cdot\sec(xy)\cdot x + \sec^2(xy)\cdot 1\right)$$

$$+ 2x\cdot\sec^2(xy)\cdot x + \tan(xy)\cdot(0)$$

$$= 2x^3y\sec^2(xy)\cdot\tan(xy) + 3x^2\sec^2(xy)$$

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$$\frac{d^2w}{dx\,dy} = \frac{d}{dx}\left(\frac{dw}{dy}\right) = \frac{d}{dx}\left(x^3\sec^2(xy)\right)$$

$$= x^3\cdot 2\sec(xy)\cdot\tan(xy)\cdot\sec(xy)\cdot y + \sec^2(xy)\cdot 3x^2$$

$$= 2x^3y\sec^2(xy)\cdot\tan(xy) + 3x^2\sec^2(xy) = \frac{d^2w}{dy\,dx} = w_{yx}$$

$$,\ \frac{d^2w}{dx^2} = w_{xx},\ \frac{d^2w}{dy^2} = w_{yy},\ \frac{d^2w}{dy\,dx} = w_{xy}\ \&\ \frac{d^2w}{dx\,dy} = w_{yx}.$$