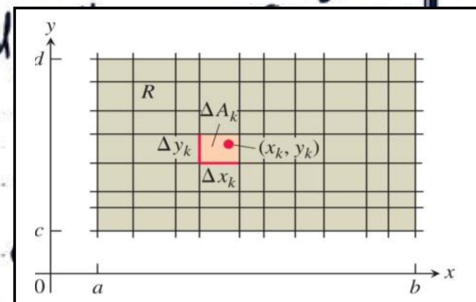
Double Integrals

Consider the simplest type of planar region a rectangle we consider a function $f(x,y)$ defined on a rectangular region R , $R: a \leq x \leq b,\ c \leq y \leq d$



- if we subdivide R into small rectangles

these lines divide R into n rectangles where the number of these n rectangles gets larges as the width and height get small.

- The small rectangular piece of width Δx and height Δy has an Area $\Delta A = \Delta x\,\Delta y$

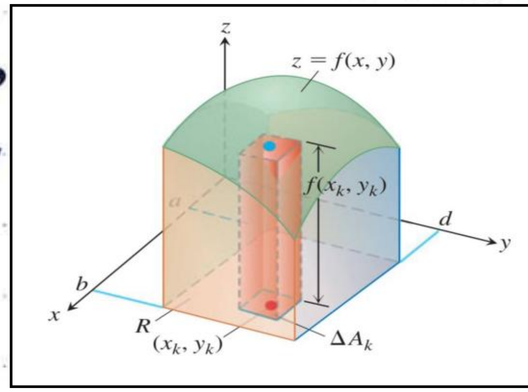
- To form a Riemann sum we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at that point by the area ΔA_k & add together the product

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

- when the limit of the sums S_n exist, then the function f is said to be integrable and the limit is called the double integral of f over R, written as

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

- when $f(x,y)$ is a positive function over a rectangular region R in the xy plane, we interpret the double integral of f over R as the volume of the 3-dimension Solid region over the xy-plane bounded below by R & above by the surface $z = f(x,y)$



- Each term $f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k
- The sum S_n thus approximates what we call the total volume of the Solid which defined to be

$$\text{Volume} = \lim_{n \to \infty} S_n = \iint_R f(x,y)\, dA$$

مهم

Where $\Delta A_k \to 0$ as $n \to \infty$

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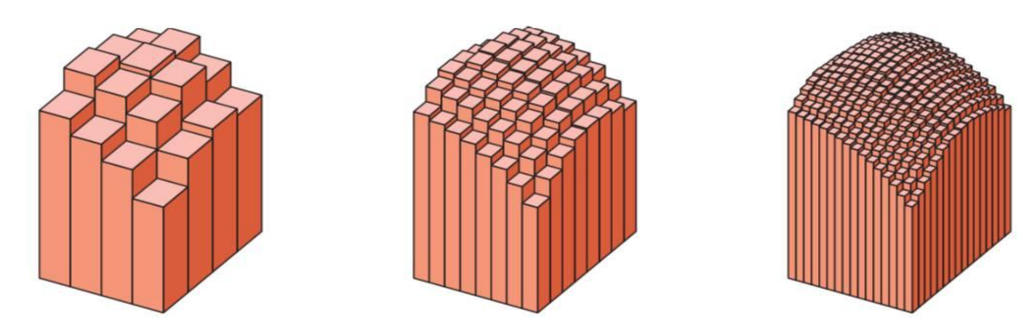
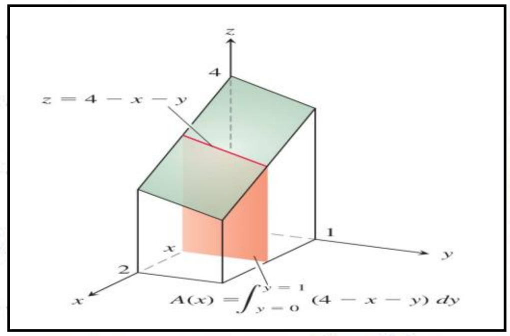


(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Fubini's Theorem for Calculating Double Integral

if we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R: 0 \le x \le 2$, $0 \le y \le 1$ in the xy-plane.



if we take slices perpendicular to the x axis, then the volume

$$V = \int_{x=0}^{x=2} A(x)\, dx$$, where $A(x)$ is the cross section area at x. We can calculate $A(x)$ as

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy$$ which is the area under the curve $z = 4 - x - y$

$$V = \int_{x=0}^{x=2} A(x)\, dx = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\, dy \right) dx$$

$$= \int_{x=0}^{x=2} 4y - xy - \frac{y^2}{2} \Big]_{y=0}^{y=1} dx = \int_{x=0}^{} \left[\left(4(1) - x(1) - \frac{(1)^2}{2}\right) - 0 \right] dx$$

$$= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx = \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 =$$

$$V = \left(\frac{7}{2}(2) - \frac{(2)^2}{2} \right) - (0) = \frac{10}{2} = 5.$$

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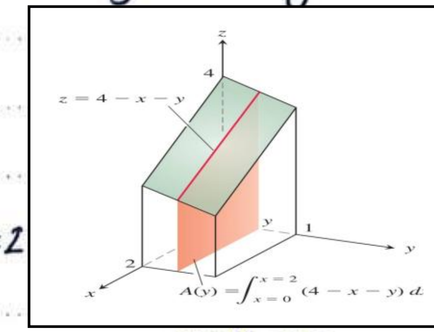
Now if we had calculated the volume by slicing with planes perpendicular to the y-axis



The typical cross sectional area is

$$A(y) = \int_{x=0}^{x=2} (4-x-y)\,dx = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2}$$

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$$= 6 - 2y$$

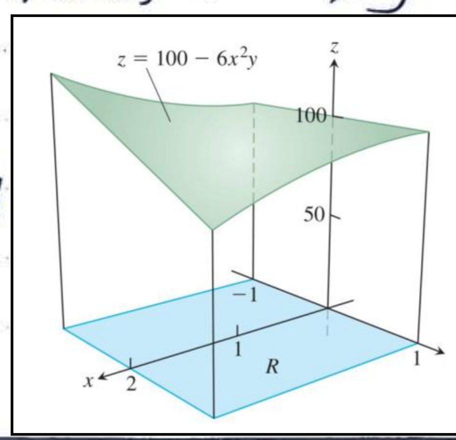
$$\therefore \text{The volume} = \int_{y=0}^{y=1} A(y)\,dy = \int_{y=0}^{y=1} (6-2y)\,dy = \left[6y - y^2\right]_0^1 = 5$$

نلاحظ هنا أن الحجم نفس المحسوب بالطريقتين لذا نستنتج Fubini's Theorem:

if $f(x,y)$ is continuous throughout the rectangular Region R $a \le x \le b$, $c \le y \le d$ then حفظ

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Ex} Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2y$ and $R: 0 \le x \le 2,\ -1 \le y \le 1$



Sol} $\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_0^2 (100 - 6x^2y)\,dx\,dy$

$$= \int_{-1}^{1} \left[100x - \frac{6x^3y}{3}\right]_0^2 dy = \int_{-1}^{1} (200 - 16y)\,dy$$

$$= \left[200y - \frac{16y^2}{2}\right]_{-1}^{1} = \left(200 - \frac{16(1)^2}{2}\right) - \left[200(-1) - \frac{16(-1)^2}{2}\right]$$

$$= 400$$

Reversing the order of the integration gives same answer

$$\int_0^2\int_{-1}^{1}(100 - 6x^2y)\,dy\,dx = \int_0^2\left[100y - 3x^2y^2\right]_{-1}^{1}dx$$

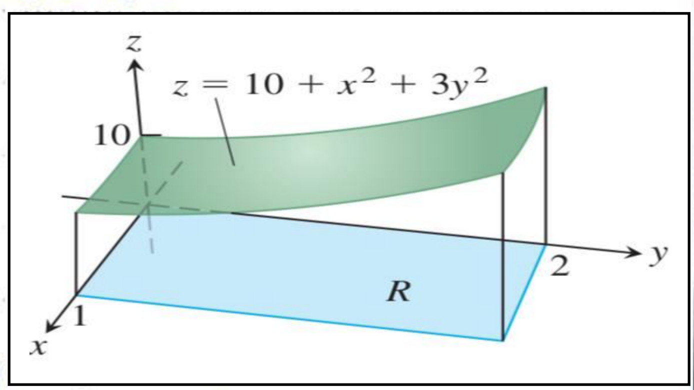
$$= \int_0^2\left[(100 - 3x^2) - (-100 - 3x^2)\right]dx$$

$$= \int_0^2(100 - \cancel{3x^2} + 100 + \cancel{3x^2})\,dx = \int_0^2 200\,dx = \left[200x\right]_0^2$$

$$= 400$$

Ex} Find the volume of the region bounded above by the ellipitical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \leq x \leq 1,\ 0 \leq y \leq 2$.

Sol} The volume is given by



$$V = \iint_R (10 + x^2 + 3y^2)\,dA =$$

$$= \int_0^1\int_0^2 (10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1 \left[10y + x^2y + y^3\right]_0^2 dx = \int_0^1 (20 + 2x^2 + 8)\,dx$$

$$= \left[20x + \frac{2}{3}x^3 + 8x\right]_0^1 = \frac{86}{3}$$

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Ex} Evaluate the integral $\int_{\pi}^{2\pi}\int_{0}^{\pi}(\sin x+\cos y)\,dx\,dy$

$$=\int_{\pi}^{2\pi}\left[-\cos x+x\cos y\right]_{0}^{\pi}dy=\int_{\pi}^{2\pi}(2+\pi\cos y)\,dy$$

$$=\left[2y+\pi\sin y\right]_{\pi}^{2\pi}=2\pi$$

Ex} Evaluate the integral $\int_{0}^{1}\int_{0}^{1}\frac{y}{1+xy}\,dx\,dy$

$$=\int_{0}^{1}\left[\ln|1+xy|\right]_{0}^{1}dy=\int_{0}^{1}\ln|1+y|\,dy$$

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$$=\left[y\ln|1+y|-y+\ln|1+y|\right]_{0}^{1}=2\ln 2-1$$

$\int \ln x\,dx = x\ln(x)-x+C$ لا تنسَ القاعدة

Ex} Evaluate the integral ① $\iint_R \frac{xy^3}{x^2+1}\,dA$, $R: 0\le x\le 1$, $0\le y\le 2$.

Sol}
$$=\int_{0}^{1}\int_{0}^{2}\frac{xy^3}{x^2+1}\,dy\,dx=\int_{0}^{1}\left[\frac{xy^4}{4(x^2+1)}\right]_{0}^{2}dx=\int_{0}^{1}\left(\frac{x(2)^4}{4(x^2+1)}-\frac{x(0)}{4(x^2+1)}\right)dx$$

$$=\int_{0}^{1}\frac{4x}{x^2+1}\,dx=\left[2\ln|x^2+1|\right]_{0}^{1}=2\ln 2$$

② $$\iint_R \frac{y}{x^2y^2+1}\,dA=\int_{0}^{1}\int_{0}^{1}\frac{y}{(xy)^2+1}\,dx\,dy=\int_{0}^{1}\left[\tan^{-1}(xy)\right]_{0}^{1}dy$$

$$=\int_{0}^{1}\tan^{-1}y\,dy=\left[y\tan^{-1}y-\frac{1}{2}\ln|1+y^2|\right]_{0}^{1}=\frac{\pi}{4}-\frac{1}{2}\ln 2$$