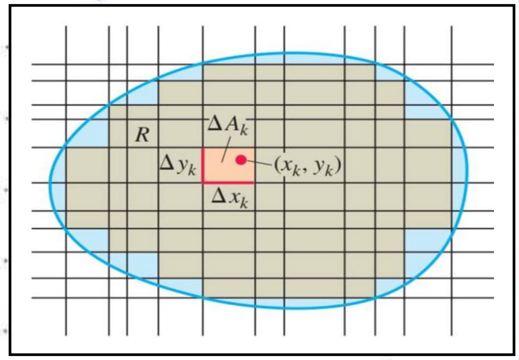
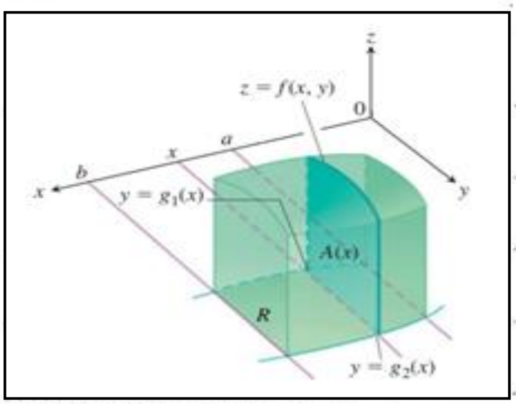
Double Integrals over General Regions

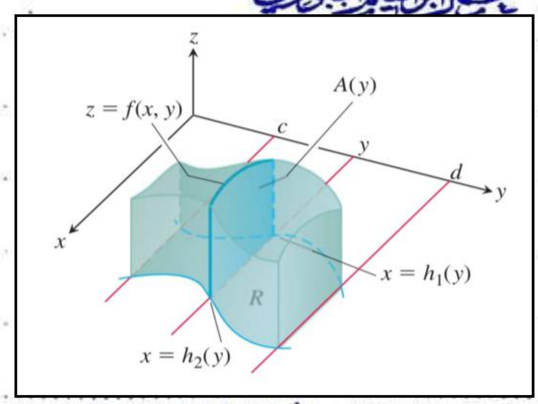
- The double integral of function $f(x,y)$ over bounded, nonrectangular region R,



- These double integrals also evaluated by $V = \iint_R f(x,y)\,dA$ with the main practical problem being that of determining the limits of integral

- Hence, the region of integration may have boundaries other than line segments, the limits of integration often involve variables, not just constants.

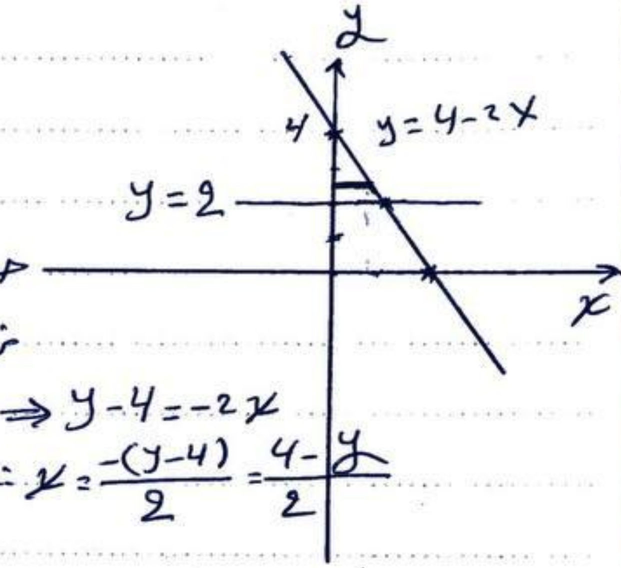




$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \qquad V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Ex} Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

① $\int_0^1 \int_2^{4-2x} dy\, dx$



$x=0\ ,\ x=1\ \&\ y=2, y=4-2x$ حدود التكامل هي

$y=4-2x$ نرسم معادلة الخط المستقيم

$y=4-2x \Rightarrow y-4=-2x$

$\therefore x=\frac{-(y-4)}{2}=\frac{4-y}{2}$

$= \int_2^4 \int_0^{\frac{4-y}{2}} dx\, dy$

② $\int_0^2 \int_0^{4-y^2} y\, dx\, dy$

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حدود التكامل

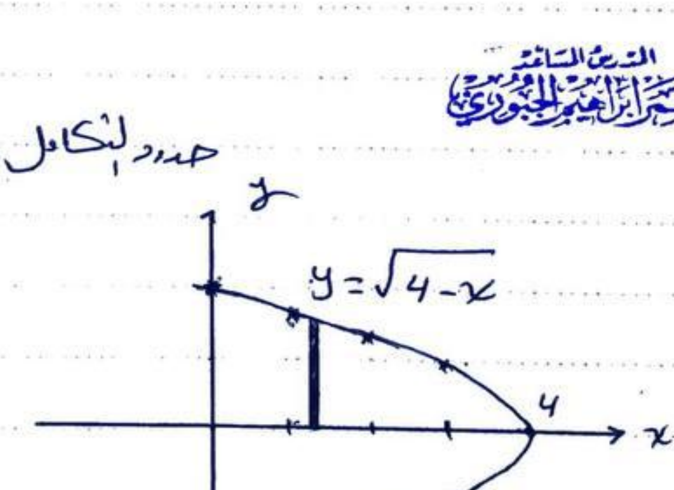
$y=0\ ,\ y=2.$

$x=0\ ,\ x=4-y^2$

$x=4-y^2 \Rightarrow y^2=4-x$

$y=\mp\sqrt{4-x}$

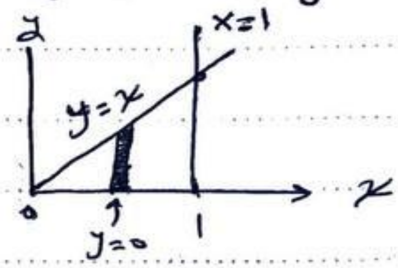
x	y
0	∓ 2
1	$\mp\sqrt{3}$
2	$\mp\sqrt{2}$
3	∓ 1
4	0



$\therefore \int_0^4 \int_0^{\sqrt{4-x}} dy\, dx$

Ex) Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis & the lines $y = x$ & $x = 1$ & whose top lies in the plane $z = 3 - x - y$.

Sol) For any x between 0 & 1



y vary from $y=0$ to $y=x$

$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{\cancel{3}x^3}{2 \times \cancel{3}}\right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2} = 1.$$

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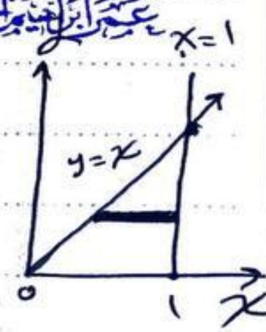
If the order of integration is reversed.



$$V = \iint_R f(x,y)\,dx\,dy = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy$$

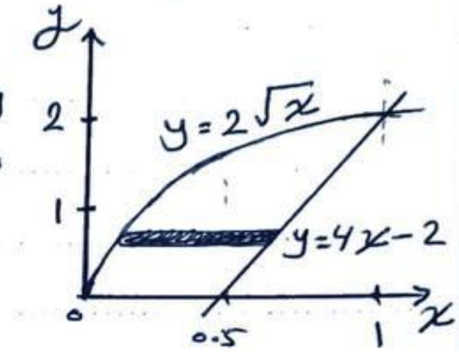
$$= \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_y^1 dy = \int_0^1 \left[3 - \frac{1}{2} - y - \left(3y - \frac{y^2}{2} - y^2\right)\right] dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - \frac{4y^2}{2} + \frac{3}{2} \cdot \frac{y^3}{3}\right]_0^1$$

$$= \frac{5}{2} - 2 + \frac{1}{2} = 1$$

Ex} Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$ and the x-axis.

Sol} الخطوة الاولى هي تحديد حدود التكامل ولفهمها نرسمها بشكل صحيح.



$$V = \iint_R (16 - x^2 - y^2)\, dA$$

$$y = 2\sqrt{x} \Rightarrow y^2 = 4x \Rightarrow x = \frac{y^2}{4}$$

$$y = 4x - 2 \Rightarrow x = \frac{y+2}{4}$$

$$\therefore V = \int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} (16 - x^2 - y^2)\, dx\, dy = \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{\frac{y^2}{4}}^{\frac{y+2}{4}} dy$$

$$= \int_0^2 \left[\left(16\left(\frac{y+2}{4}\right) - \frac{1}{3}\left(\frac{y+2}{4}\right)^3 - y^2 \cdot \frac{y+2}{4}\right) - \left(16 \cdot \frac{y^2}{4} - \frac{1}{3}\left(\frac{y^2}{4}\right)^3 - y^2\left(\frac{y^2}{4}\right)\right)\right] dy$$

$$= \int_0^2 \left(4(y+2) - \frac{(y+2)^3}{192} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{192} + \frac{y^4}{4}\right) dy$$

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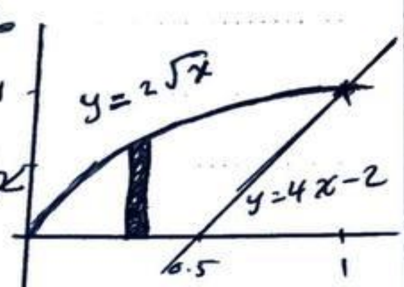
$$= \int_0^2 \left(4(y+2) - \frac{(y+2)^3}{192} - \frac{y^3}{4} - \frac{y^2}{2} - 4y^2 + \frac{y^6}{192} + \frac{y^4}{4}\right) dy$$

$$= \left[\frac{\cancel{4}^{2}(y+2)^2}{\cancel{2}} - \frac{(y+2)^4}{4 \times 192} - \frac{y^4}{4 \times 4} - \frac{9y^3}{2 * 3} + \frac{y^7}{192 * 7} + \frac{y^5}{4 \times 5}\right]_0^2$$

$$= \left(2(2+2)^2 - \frac{(2+2)^4}{4 \times 192} - \frac{2^4}{16} - \frac{9}{6}2^3 + \frac{2^7}{192 \times 7} + \frac{2^5}{20}\right) - \left(2^3 - \frac{2^4}{4 \times 192}\right)$$

$$= 12.38$$

يمكن حل المثال بطريقة أخرى من خلال أخذ التكامل $(dy\,dx)$ وعليه يمكن ملاحظة أن حدود التكامل المختارة بهذه الطريقة تقسم المنطقة الى منطقتين وهما



$$V=\int_0^{0.5}\int_0^{2\sqrt{x}}(16-x^2-y^2)\,dy\,dx+\int_{0.5}^{1}\int_{4x-2}^{2\sqrt{x}}(16-x^2-y^2)\,dy\,dx$$

$$=\int_0^{0.5}\left[16y-x^2y-\frac{y^3}{3}\right]_0^{2\sqrt{x}}dx+\int_{0.5}^{1}\left[16y-x^2y-\frac{y^3}{3}\right]_{4x-2}^{2\sqrt{x}}dx$$

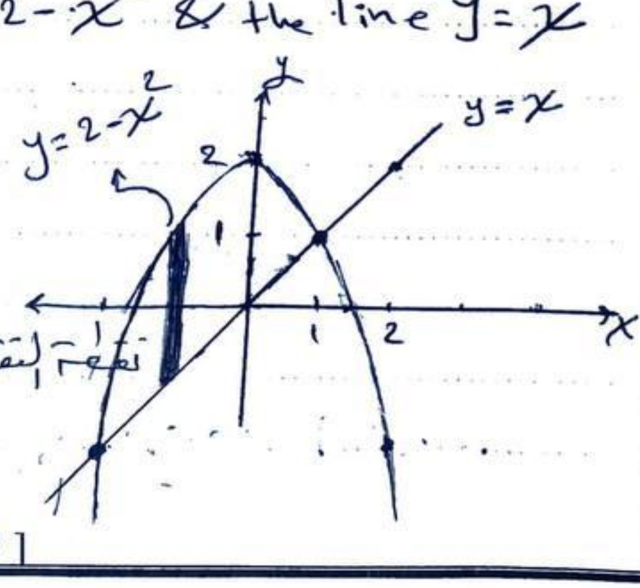
$$=\int_0^{0.5}\left(32\sqrt{x}-2x^{\frac{5}{2}}-\frac{8x^{\frac{3}{2}}}{3}\right)dx+\int_{0.5}^{1}\left(\left(32\sqrt{x}-2x^{\frac{5}{2}}-\frac{8x^{3/2}}{3}\right)-\left(16(4x-2)-x^2(4x-2)-\frac{(4x-2)^3}{3}\right)\right)dx.$$

ويكمل بقية الحل كما متعارف عليه

Ex] Find the volume of the Solid that is bounded above by the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ & the line $y=x$ in the xy-plane.

Sol]

$$V=\iint_R x^2\,dy\,dx=$$



نقطة التقاطع من خلال الرسم هي $(1,1)$, $(-2,-2)$ أو من خلال

$$2-x^2=x\Rightarrow x^2+x-2=0\Rightarrow$$

$$(x+2)(x-1)=0$$

either $x+2=0 \Rightarrow x=-2 \Rightarrow y=-2$

or $x-1=0 \Rightarrow x=1 \Rightarrow y=1$.

$$V=\int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx=\int_{-2}^{1}\left[x^2 y\right]_{x}^{2-x^2}dx$$

$$=\int_{-2}^{1}\left(x^2(2-x^2)-x^2\cdot x\right)dx=\int_{-2}^{1}\left(2x^2-x^4-x^3\right)dx$$

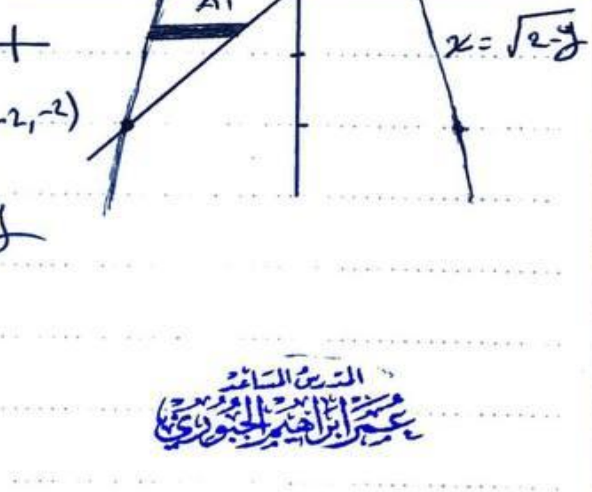
$$=\left[2\cdot\frac{x^3}{3}-\frac{x^5}{5}-\frac{x^4}{4}\right]_{-2}^{1}=\left(\frac{2}{3}-\frac{1}{5}-\frac{1}{4}\right)-\left(\frac{-16}{3}+\frac{32}{5}-\frac{16}{4}\right)$$

$$=\frac{63}{20}$$

أيضاً يمكن حل المثال بطريقة أخرى $(dx\,dy)$

وهنا يكون لدينا مساحات مقسمة

$y=2-x^2 \Rightarrow x^2=2-y \Rightarrow x=\mp\sqrt{2-y}$



$$V=\int_{-2}^{0}\int_{-\sqrt{2-y}}^{y}x^2\,dx\,dy+\int_{0}^{2}\int_{-\sqrt{2-y}}^{0}x^2\,dx\,dy+$$

$$\int_{0}^{1}\int_{0}^{y}x^2\,dx\,dy+\int_{1}^{2}\int_{0}^{\sqrt{2-y}}x^2\,dx\,dy$$

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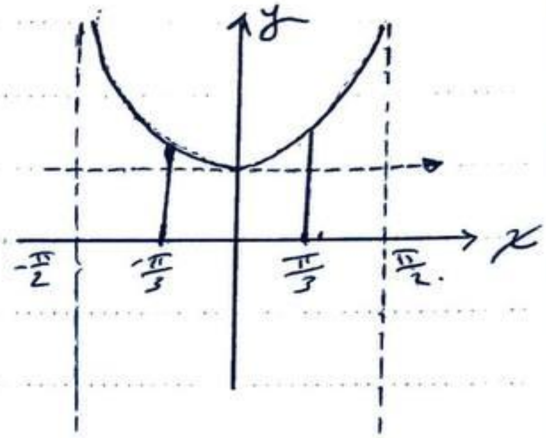
Ex] Find the volume of the Solid bounded on the front and back by the planes $x = \pm\frac{\pi}{3}$, on the sides by cylinders $y = \pm \sec x$, above by cylinder $Z = 1 + y^2$ & below by the xy-plane.



Sol] $V = \iint_R (1+y^2)\, dy\, dx$

$$V = 4\int_0^{\frac{\pi}{3}}\int_0^{\sec x} (1+y^2)\, dy\, dx$$

$$= 4\int_0^{\frac{\pi}{3}} y + \frac{y^3}{3}\Big]_0^{\sec x} dx = 4\int_0^{\frac{\pi}{3}} \left(\sec x + \frac{\sec^3 x}{3}\right) dx$$

$$= 4\left[\ln|\sec x + \tan x| + \frac{1}{3}\times\frac{1}{2}\left(\sec x \tan x + \mathrm{Log}(\sec x + \tan x)\right)\right]_0^{\frac{\pi}{3}}$$

$$= 4\Big[\ln\left|\sec\tfrac{\pi}{3} + \tan\tfrac{\pi}{3}\right| + \frac{1}{6}\left(\sec\tfrac{\pi}{3}\cdot\tan\tfrac{\pi}{3} + \log\left(\sec\tfrac{\pi}{3} + \tan\tfrac{\pi}{3}\right)\right) -$$

$$\left[\ln|\sec 0 + \tan 0| + \frac{1}{6}\left(\sec 0\cdot\tan 0 + \mathrm{Log}(\sec 0 + \tan 0)\right)\right]\Big]$$

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