Directional Derivatives

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The function $f(x,y)$ is defined throughout a region R & $P_o(x_o,y_o)$ is a point in R & that $u = u_1 i + u_2 j$ is a unit vector, hence the derivative of f at point $P_o(x_o,y_o)$ in the direction of the unit vector u is $(D_u f)_{P_o}$ (The derivative of f at P_o in the direction of u)

- For a physical intepetation of the directional derivative, suppose that $T = f(x,y)$ is the temperature at each point (x,y) over a region in the plane. Then, $f(x_o,y_o)$ is the temperature at the point (x_o,y_o) and $(D_u f)_{P_o}$ is the instantaneous rate of change of the temperature at P_o stepping off in the direction u

- The gradient vector (gradient) of $f(x,y)$ at a point $P_o(x_o,y_o)$ is the vector

$$\boxed{\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j}$$

حفظ

- The notation ∇f is read "grad f", "gradient f" & "del f"

- The directional derivative is a Dot product :- if $f(x,y)$ is differentiable in an open region containing $P_o(x_o,y_o)$, then

$$\boxed{(D_u f)_{P_o} = \left(\frac{df}{ds}\right)_{u,P_o} = (\nabla f)_{P_o} \cdot u}$$

حفظ

Ex] Find the derivative of $f(x,y) = x \cdot e^{y} + \cos(xy)$ at the point $(2,0)$ in the direction of $V = 3i + 4j$

Sol]

في البداية يجب ان نستخرج unit vector من خلال

$$u = \frac{V}{|V|} = \frac{3i - 4j}{\sqrt{3^2 + 4^2}} = \frac{3i - 4j}{5} = \frac{3}{5} i - \frac{4}{5} j$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j \quad , \quad \frac{\partial f}{\partial x} = f_x = e^{y} - y \sin(xy)$$

$$\left. \frac{\partial f}{\partial x} \right|_{(2,0)} = e^{0} - (0) \sin(2 \times 0) = 1 - 0 = 1$$

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$$\left. \frac{\partial f}{\partial y} \right|_{(2,0)} = f_y(2,0) = \left[x e^{y} - x \sin(xy) \right] = 2e^{0} - 2(0) = 2$$

$\therefore$ The gradient $\nabla f|_{(2,0)} = 1i + 2j$

$$\therefore (D_u f)_{(2,0)} = \nabla f|_{(2,0)} \cdot u = (i + 2j) \cdot \left(\frac{3}{5} i - \frac{4}{5} j \right)$$

$$= \frac{3}{5} - \frac{8}{5} = -1$$

- Functions of Three Variables

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$$ حفظ

where f is function of x, y, z and a unit vector $u = u_1 i + u_2 j + u_3 K$, then

$$D_u f = \nabla f \cdot u = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} \cdot u_2 + \frac{\partial f}{\partial z} u_3$$ حفظ

Ex} Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $v = 2i - 3j + 6K$

unit vector u (direction) $= \dfrac{v}{|v|} = \dfrac{2i - 3j + 6K}{\sqrt{2^2 + 3^2 + 6^2}} = \dfrac{2i - 3j + 6K}{7}$

$$u = \frac{2}{7} i - \frac{3}{7} j + \frac{6}{7} K$$

The partial derivative, $\dfrac{\partial f}{\partial x} = (3x^2 - y^2)\Big|_{(1,1,0)} = 3(1)^2 - (1)^2 = 2.$

$$\frac{\partial f}{\partial y} = -2xy\Big|_{(1,1,0)} = -2(1)(1) = -2$$

$$\frac{\partial f}{\partial z} = -1\Big|_{(1,1,0)} = -1, \quad \nabla f\Big|_{(1,1,0)} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$$

$$\therefore \nabla f\Big|_{(1,1,0)} = 2i - 2j - K, \quad (D_u f)_{(1,1,0)} = \nabla f\Big|_{(1,1,0)} \cdot u$$

$$\therefore (D_u f)_{(1,1,0)} = (2i - 2j - K) \cdot \left(\frac{2}{7} i - \frac{3}{7} j + \frac{6}{7} K\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$$

Algebra Rules for Gradients.

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1- Sum Rules - $\nabla(f + g) = \nabla f + \nabla g$

2- Difference Rules $\nabla(f - g) = \nabla f - \nabla g$

3- Constant Multiple Rule: $\nabla(Kf) = K \nabla f$

4- Product Rule: $\nabla(fg) = f \nabla g + g \nabla f$

5- Quotient Rule: $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g \nabla f - f \nabla g}{g^2}$

Ex} Find the rate of change of $f(x,y,z) = x^2 y z^3$ at $P_0(2,1,-1)$ in the direction of x-coordinate, $V = i - 2j + 3K$ direction & direction from P_0 to $P_1(5,2,4)$.

Sol} المطلوب في السؤال المشتقة الاتجاهية وبالاتجاهات المحددة في السؤال، وهنا نستخرج ∇f لانها لا تتغير مع الاتجاه ثم نكمل باقي الحل كما هو مطلوب

$$\nabla f \Big|_{(2,1,-1)} = \left(\frac{df}{dx} i + \frac{df}{dy} j + \frac{df}{dz} K\right)_{(2,1,-1)}$$

$$\frac{df}{dx} = y z^3 * 2x \Big|_{(2,1,-1)} = (1)(-1)^3 * 2(2) = -4$$

$$\frac{df}{dy} = x^2 z^3 \Big|_{(2,1,-1)} = (2)^2 (-1)^3 = -4$$

$$\frac{df}{d} = x^2 y * 3z^2 \Big|_{(2,1,-1)} = (2)^2 (1) * 3(-1)^2 = 12$$

$$\therefore \nabla f = -4i - 4j + 12K.$$

- in x - direction $\Rightarrow \vec{V} = i \Rightarrow u = \dfrac{i}{\sqrt{1}} = i$

$$\therefore D_u f \Big)_{(2,1,-1)} = \nabla f \cdot u = (-4i - 4j + 12k) \cdot (i + 0j + 0k) = -4$$

- in direction of $v = i - 2j + 3K \Rightarrow u = \dfrac{i - 2j + 3K}{\sqrt{1^2 + 2^2 + 3^2}} = \dfrac{1}{\sqrt{14}} i - \dfrac{2}{\sqrt{14}} j + \dfrac{3}{\sqrt{14}} K$

$$D_u f \Big)_{(2,1,-1)} = (-4i - 4j + 12K) \cdot \left(\frac{1}{\sqrt{14}} i - \frac{2}{\sqrt{14}} + \frac{3}{\sqrt{14}} K\right) = \frac{1}{\sqrt{14}} (-4 + 8 + 36)$$

- in the direction from P_0 to P_1: $V = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)K$

$$V = (5-2)i + (2-1)j + (4 - -1)K = 3i + j + 5K.$$

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$$u = \frac{v}{|v|} = \frac{3i + j + 5K}{\sqrt{3^2 + 1^2 + 5^2}} = \frac{3}{\sqrt{35}} i + \frac{1}{\sqrt{35}} j + \frac{5}{\sqrt{35}} K.$$

$$D_u f \Big|_{(2,1,-1)} = \nabla f \cdot u = (-4i - 4j + 12K) \cdot \left(\frac{3}{\sqrt{35}} i + \frac{1}{\sqrt{35}} j + \frac{5}{\sqrt{35}} K\right)$$

$$= \left((-4)\left(\frac{3}{\sqrt{35}}\right) - (4) \cdot \frac{1}{\sqrt{35}} + 12 \cdot \frac{5}{\sqrt{35}}\right) = \frac{1}{\sqrt{35}} \left(-4 \times 3 - 4 \times 1 + 12 \times 5\right)$$

$$= \frac{1}{\sqrt{35}} \times 44$$

Tangent plane & Normal Line

Tangent plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t \quad \& \quad z = z_0 + f_z(P_0)t$$

Ex] Find the tangent plane and normal line of the surface

$f(x, y, z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1, 2, 4)$

Sol] $f_x = \frac{df}{dx} = 2x, \quad f_y = \frac{df}{dy} = 2y \quad \& \quad f_z = 1$

$\therefore$ Tangent plane $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$\therefore \quad f_x \Big|_{P_0} = 2(1) = 2, \quad f_y \Big|_{P_0} = 2 \times 2 = 4, \quad f_z \Big|_{P_0} = 1$

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$\therefore 2(x-1)+4(y-2)+(z-4)=0$

$\therefore 2x-2+4y-8+z-4=0 \Rightarrow \boxed{2x+4y+z=14}$ Tangent plane.

The normal Line $x=x_0+f_x(P_0)\,t=1+2t$

$y=y_0+f_y(P_0)\,t \Rightarrow y=2+4t$.

$z=z_0+f_z(P_0)\,t \Rightarrow z=4+t$

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Ex) Find the tangent plane & normal line of the surface $\cos\pi x - x^2y + e^{xz} + yz = 4$ at the point $P_0(0,1,2)$

Sol) $f_x\Big|_{P_0} = \dfrac{df}{dx}\Big|_{P_0} = (-\sin\pi x)\pi - 2yx + e^{xz}\cdot z\Big|_{P_0}$

$f_x\Big|_{P_0} = -\pi\sin(\pi * 0) - 2(1)(0) + 2e^{(0)(2)} = 2$

$f_y\Big|_{P_0} = \dfrac{df}{dy}\Big|_{P_0} = -x^2 + z\Big|_{P_0} = -(0)^2 + (2) = 2$

$f_z\Big|_{P_0} = \dfrac{df}{dz}\Big|_{P_0} = e^{xz}\cdot x + y\Big|_{P_0} = e^{(0)(2)}\cdot(0)+1 = 1$

$\therefore$ Tangent plane $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

$\therefore 2(x-0)+2(y-1)+1(z-2)=0 \Rightarrow 2x+2y+z=4$

Normal Line. $x=x_0+f_x(P_0)\,t=0+2t \Rightarrow x=2t$

$y=y_0+f_y(P_0)\,t=1+2t$, $z=z_0+f_z(P_0)\,t=2+t$

Linearization

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The linearization of a function $f(x,y)$ at a point (x_0, y_0) where f is differentiable is a function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

hence the approximation $f(x,y) \approx L(x,y)$

Ex} Find the approximation of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point $(3,2)$

Sol} we have that $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$$f(x_0,y_0) = f(3,2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = 3^2 - 3\times 2 + \frac{1}{2}(2)^2 + 3 = 8$$

$$f_x(3,2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (2x - y)_{(3,2)} = 2\times 3 - 2 = 4$$

$$f_y(3,2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (-x + y)_{(3,2)} = -3 + 2 = -1$$

$$\therefore L(x,y) = 8 + 4(x-3) + (-1)(y-2) = 8 + 4x - 12 - y + 2$$

$L(x,y) = 4x - y - 2$ The linearization of f at $(3,2)$.

* For functions of more than two variables, the linearization of $f(x,y,z)$ at a point $P_0(x_0,y_0,z_0)$

$$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$

Ex} Find the linearization $L(x,y,z)$ of.
$f(x,y,z) = x^2 - xy + 3\sin z$ at the point $(2,1,0)$

Sol}

$$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$

$$f(P_0) = (x^2 - xy + 3\sin z)_{(2,1,0)} = 2^2 - 2(1) + 3\sin(0) = 2$$

$$f_x = \frac{d}{dx}(x^2 - xy + 3\sin z)_{(2,1,0)} = (2x - y)_{(2,1,0)} = 2(2) - 1 = 3$$

$$f_y = \frac{d}{dy}(x^2 - xy + 3\sin z)_{(2,1,0)} = (-x)_{(2,1,0)} = -2$$

$$f_z = \frac{d}{dz}(x^2 - xy + 3\sin z)_{(2,1,0)} = 3\cos z = 3\cos(0) = 3$$

$$\therefore L(x,y,z) = 2 + 3(x-2) + -2(y-1) + 3(z-0)$$

$$= 2 + 3x - 6 - 2y + 2 + 3z$$

$$= 3x - 2y + 3z - 2.$$

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The total differential

when we move from (x_0, y_0) to a point (x_0+dx, y_0+dy) the resulting change $df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$ حفظ

also for functions with three variables حفظ

$$df = f_x(x_0,y_0,z_0)\,dx + f_y(x_0,y_0,z_0)\,dy + f_z(x_0,y_0,z_0)\,dz.$$

Ex} if $w = x^3 \sin y$. Find the differential dw

$$dw = w_x\,dx + w_y\,dy = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

$$dw = \frac{\partial}{\partial x}(x^3 \sin y)\,dx + \frac{\partial}{\partial y}(x^3 \sin y)\,dy$$

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$$dw = 3x^2 \sin y\,dx + x^3 \cos y\,dy$$

Ex} If $z = f(x,y) = x^2 + 3xy - y^2$, Find the differential dz

then if x changes from 2 to 2.05 & y changes from 3 to 2.96 Find values of Δz and dz.

Sol}

$$dz = z_x\,dx + z_y\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = \frac{\partial}{\partial x}(x^2+3xy-y^2)\,dx + \frac{\partial}{\partial y}(x^2+3xy-y^2)\,dy$$

$$dz = (2x+3y)\,dx + (3x-2y)\,dy$$

$$dx = \Delta x = 2.05 - 2 = 0.05 \quad , \quad dy = \Delta y = 2.96 - 3 = -0.04$$

$$x_0 = 2 \quad , \quad y_0 = 3$$

$$dz = [2(2)+3(3)] * 0.05 + [3(2)-2(3)] * -0.04 = 0.65$$

$$\Delta z = z_2 - z_1 = f(2.05, 2.96) - f(2,3)$$

$$= [(2.05)^2 + 3(2.96)(2.05) - 2.96^2] - [2^2 + 3(2)(3) - 3^2]$$

$$= 0.6449$$

Ex} For a cylindrical can is designed to have a radius of 1 inch & a height of 5 inches, but the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Sol} we have that $V = \pi r^2 h$, $r_0 = 1$ in, $h_0 = 5$ in

$$dV = V_r(r_0, h_0)\,dr + V_h(r_0, h_0)\,dh$$

$$= \frac{dV}{dr}dr + \frac{dV}{dh}dh = \frac{d}{dr}(\pi r^2 h)dr + \frac{d}{dh}(\pi r^2 h)\,dh.$$

$$dV = 2\pi r h\Big|_{(r_0, h_0)} dr + \pi r^2\Big|_{(r_0, h_0)} dh$$

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$$= 2\pi(1)(5)(+0.03) + \pi(1)^2(-0.01) = 0.2\pi \simeq 0.63 \text{ in}^3$$

Ex} For a right circular cylindrical storage tank that are 25 ft high with a radius of 5 ft. How sensitive are the tanks' volumes to small variations in height.

Sol} $$dV = 2\pi r h\Big|_{(r_0, h_0)} dr + \pi r^2\Big|_{(r_0, h_0)} dh \quad , \quad r_0 = 5 \text{ ft}, h_0 = 25 \text{ ft}$$

$$= 2\pi(5)(25)\,dr + \pi(5)^2\,dh$$

$$dV = 250\pi\,dr + 25\pi\,dh$$

ومن هذه المعادلة نستنتج إنه إذا تغير نصف القطر بمقدار واحد فإن الحجم سيتغير بمقدار 250π وعندما يتغير الارتفاع بمقدار واحد فإن الحجم سيتغير بمقدار 25π لذا فإن تأثير تغير نصف القطر أكبر من تأثير تغير الارتفاع بنفس المقدار.