General Differential Equations.

- In general, a differential equation is an equation that Contains an unknown function and one or more of it's derivatives.
- The order of differential equation is the order of the highes derivative that occurs in the equation

examples of first order differential equations (Contains only first order derivative)

$$\frac{dp}{dt}=kp \,,\quad \frac{dy}{dx}=x^2+y \,,\quad \frac{ds}{dt}=5t+12 \,,\quad \frac{dy}{dx}=\frac{xy}{x^2+4}$$

- Examples of Second order differential equations (contains Second order derivative)

$$\frac{d^2x}{dt^2}=cx \,,\quad \frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0 \,,\quad \frac{d^2y}{dx^2}+\frac{dy}{dx}-2=x^2 .$$

- Examples of n-th order differential equations

$$ay^{n}+by'+cy=G(x) \,,\quad y^{n}+4y=e^{3x} .$$

First order differential equations

First order differential equations can be devided into Five types

① Separable equations   معادلات قابلة للفصل

في هذا النوع يمكن فصل x و dx بجهة و y و dy في جهة أخرى.

The general formula $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

$\therefore \boxed{h(y)\,dy = g(x)\,dx}$ وبعد كتابة المعادلة بهذه الصيغة يمكن ان نكامل الطرفين بصورة اعتيادية كما في المعادلة التالية.

Ex Solve the equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$

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sol $y^2\,dy = x^2\,dx$

$\therefore \int y^2\,dy = \int x^2\,dx \Rightarrow \dfrac{y^3}{3} = \dfrac{x^3}{3} + C \longrightarrow$ لانه تكامل غير محدد

$\therefore \boxed{y = \sqrt[3]{x^3 + 3C}}$ general solution

ويمكن ايجاد particular solution عند وجود initial condition كما تعلمنا في الفصل الاول.

if I.C $y(0) = 2$

$2 = \sqrt[3]{(0)^3 + 3C} \Rightarrow \sqrt[3]{3C} = 2$ بالتكعيب $\Rightarrow C = \dfrac{8}{3}$

$\therefore y = \sqrt[3]{x^3 + 8}$

Ex} Solve the differential equation. $y' = \dfrac{6x^2}{2y+\cos y}$.

Sol} $\dfrac{dy}{dx} = \dfrac{6x^2}{2y+\cos y} \Rightarrow (2y+\cos y)\,dy = 6x^2\,dx$

by $\int \Rightarrow \int (2y+\cos y)\,dy = \int 6x^2\,dx$

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$\therefore \dfrac{2y^2}{2} + \sin y = \dfrac{6x^3}{3} + c \Rightarrow y^2 + \sin y = 2x^3 + c$

Ex} Solve the differential equation. $y' = x^2 y$.

Sol} $\dfrac{dy}{dx} = x^2 y \Rightarrow \dfrac{1}{y}\,dy = x^2\,dx \longrightarrow$ by $\int$

$\therefore \int \dfrac{1}{y}\,dy = \int x^2\,dx \Longrightarrow \ln|y| = \dfrac{x^3}{3} + c$

لا تنس القيمة المطلقة لأن $\ln$ العدد السالب غير معرف.

$|y| = e^{\ln|y|} = e^{(x^3/3 + c)} \Rightarrow |y| = e^{x^3/3} \cdot e^{c}$

$\therefore y = \mp A\, e^{x^3/3}$

لا تنس القاعدة عند ضرب الدوال الأسية
تجمع الأسس

هنا فرضنا أن $A = e^{c}$ لتسهيل التعبير عن الثابت وأيضاً ممكن أن نفرض

أن $A = \mp e^{c}$ لذا يكون الحل

$y = A e^{x^3/3}$

② Homogeneous equations

The general formula is $y' = f(x, y)$

The first order diff. eq. is homogeneous if $f(tx, ty) = f(x,y)$

where t any real number $t \neq 0$

The solution done by changing variable defined by

$y = vx \Rightarrow v = \frac{y}{x}$ & $\frac{dy}{dx} = v + x \cdot \frac{dv}{dx} \rightarrow$ مشتقة حاصل ضرب دالتين

والغرض من هذا التغير هو تحويل المعادلة الى معادلة من النوع الاول (قابلة للفصل) وعليه يكون

$$\frac{dy}{dx} = f(x, y) \Longrightarrow v + x\frac{dv}{dx} = f(1, v)$$

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Ex} Find the general solution for the following 1st order diff. eqs.

① $\frac{dy}{dx} = \frac{y+x}{x}$

الخطوة الاولى نعمل الاختبار لنتأكد ما نوع المعادلة من خلال ضرب أو تحويل الدالة ($f(x,y) \Rightarrow f(tx, ty)$)

$$\frac{ty+tx}{tx} = \frac{t(x+y)}{tx} = \frac{y+x}{x} \Longrightarrow \text{Homogeneous}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} + 1 \quad \text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}, \; v = \frac{y}{x}$$

$$\therefore \cancel{v} + x\frac{dv}{dx} = \cancel{v} + 1 \Rightarrow x\frac{dv}{dx} = 1 \Rightarrow dx = x\,dv$$

$$\therefore \frac{1}{x}dx = dv \Rightarrow \text{by} \int \Rightarrow \ln x = v + c \Rightarrow \ln x = \frac{y}{x} + c$$

الآن يجب أن نكتب المعادلة بدلالة y

$$\therefore \frac{y}{x} = \ln x - C \Rightarrow y = x(\ln x + C)$$

(2) $$\frac{dy}{dx} = \frac{y}{x - 2\sqrt{xy}}$$

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$$\frac{ty}{tx - 2\sqrt{tx \cdot ty}} = \frac{ty}{t(x - 2\sqrt{xy})} = \frac{y}{x - 2\sqrt{xy}} \quad \therefore \text{Homogeneous.}$$

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{vx}{x - 2\sqrt{x \cdot vx}} = \frac{v\cancel{x}}{\cancel{x}(1 - 2\sqrt{v})} = \frac{v}{1 - 2\sqrt{v}}$$

$$\therefore v + x\frac{dv}{dx} = \frac{v}{1 - 2\sqrt{v}} \Rightarrow x\frac{dv}{dx} = \frac{v}{1 - 2\sqrt{v}} - v$$

$$x\frac{dv}{dx} = \frac{v - v(1 - 2\sqrt{v})}{1 - 2\sqrt{v}} = \frac{\cancel{v} - \cancel{v} + 2v\sqrt{v}}{1 - 2\sqrt{v}}$$

توزيع المقام

$$\therefore \frac{dx}{x} = \frac{1 - 2\sqrt{v}}{2v\sqrt{v}}\, dv \Rightarrow \frac{dx}{x} = \left(\frac{1}{2v\sqrt{v}} - \frac{1}{2v}\right) dv$$

$$\frac{dx}{x} = \left(\frac{1}{2v^{3/2}} - \frac{1}{2v}\right) dv \Rightarrow \ln x = \frac{1}{2} \cdot \frac{v^{-1/2}}{\frac{-1}{2}} - \frac{1}{2}\ln v + C$$

$$\ln x = \frac{-1}{\sqrt{v}} - \frac{1}{2\ln v} + C \Rightarrow \ln x = \frac{-1}{\sqrt{\frac{y}{x}}} - \frac{1}{2\ln\frac{y}{x}} + C$$

③ $(x-2y)\,dx + x\,dy = 0$.

الافضل عند بدء الحل نكتب المعادلة بصيغة $\frac{dy}{dx} =$

$\frac{dy}{dx} = \frac{2y-x}{x}$, Test $\frac{2yt-xt}{xt} = \frac{\cancel{t}(2y-x)}{x\cancel{t}} = \frac{2y-x}{x}$ ∴ Homogen.

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} =$

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$\frac{2y-x}{x} = \frac{2y}{x} - 1 = 2v - 1 \Rightarrow v + x\frac{dv}{dx} = 2v - 1$

$x\frac{dv}{dx} = v - 1 \Longrightarrow \frac{dx}{x} = \frac{dv}{1-v}$ by $\int \Rightarrow \ln x = -\ln(1-v) + \ln c$

لاحظ هنا الاشارة السالبة جاءت من توفير مشتقة الدالة وكتبنا الثابت بصيغة $\ln$ لكي تكون الدالة بصورة مبسطة عند حذف $\ln$.

$\ln x = \ln\frac{c}{1-v}$ by $e \Rightarrow x = \frac{c}{1-v}$

$\therefore x(1-v) = c \Rightarrow x - y = c \Rightarrow y = x - c$ General solution

④ $\left(x^2\sin\frac{y^2}{x^2} - 2y^2\cos\frac{y^2}{x^2}\right)dx + \left(2xy\cos\frac{y^2}{x^2}\right)dy = 0$

$\therefore \frac{dy}{dx} = \frac{2y^2\cos\frac{y^2}{x^2} - x^2\sin\frac{y^2}{x^2}}{2xy\cos\frac{y^2}{x^2}}$ Homogeneous 1st order.

$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{2v^2\cancel{x^2}\cos v^2 - \cancel{x^2}\sin v^2}{2\cancel{x}v\cancel{x}\cos v^2}$

$y = \cancel{x}v$ من الفرضية

من خلال تعويض عن كل $y = vx$

$$v + x\frac{dv}{dx} = \frac{2v^2\cdot\cos v^2 - \sin v^2}{2v\cos\cdot v^2}$$

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} - \frac{1}{2v}\cdot\frac{\sin v^2}{\cos v^2} \Rightarrow \frac{dx}{x} = -2v\frac{\cos v^2}{\sin v^2}dv$$

$$\frac{dx}{x} = \frac{-2v\cos v^2}{\sin v^2}\,dv \quad \text{by} \int$$

تعتبر دالة بالمقام والمشتقة متوفرة في البسط

$$\ln|x| = -\ln|\sin v^2| + \ln c$$

$$\ln|x| = \ln\frac{c}{\sin v^2} \quad \text{by } e \Rightarrow x = \frac{c}{\sin v^2}$$

$$\therefore x\sin\frac{y^2}{x^2} = c \quad \text{The general solution.}$$

(5) $(x^2+3y^2)dx = 2xy\,dy$

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$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}, \quad \text{Test} \quad \frac{t^2x^2+3t^2y^2}{2xtyt} = \frac{x^2+3y^2}{2xy} \quad \text{Homo.}$$

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{x^2+3y^2}{2xy}.$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x^2}+3v^2\cancel{x^2}}{2\cancel{x}v\cancel{x}} \Rightarrow v + x\frac{dv}{dx} = \frac{1+3v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v} \Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\frac{dx}{x} = \frac{2v}{1+v^2} \quad \text{by} \int \Rightarrow \ln x = \ln(1+v^2) + \ln c$$

$$\text{by } e \Rightarrow x = c(1+v^2) \Rightarrow x = c\left(1+\frac{y^2}{x^2}\right) \Rightarrow x^3 = cx^2 + y^2$$

$$y^2 = x^2(x - c)$$

(3) Exact Differential Equations.

A differential equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

it exact eq if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

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Steps for solving an Exact D.E.

1- Match the eq. to the form $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$ to identify $\dfrac{\partial f}{\partial x}$ & $\dfrac{\partial f}{\partial y}$.

2- Integrate $\dfrac{\partial f}{\partial x}$ with respect to x, writing the constant of integration as $C(y)$

3- Differentiate with respect to y & set the result to equal $\dfrac{\partial f}{\partial y}$ to find $C'(y)$

4- Integrate to find $C(y)$ & determine $f(x,y)$

5- write the Solution of Equation as $f(x,y) = C$

والامثلة التالية توضح خطوات الحل.

Ex] Solve the following D.E.

① $2xy\,dx + (x^2-1)\,dy = 0$

Sol] $M(x,y) = 2xy$, $N(x,y) = (x^2-1)$

$$\frac{\partial M}{\partial y} = 2x \ , \ \frac{\partial N}{\partial x} = 2x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact D.E.}$$

$\therefore$ * $\frac{\partial f}{\partial x} = 2xy$ & $\frac{\partial f}{\partial y} = (x^2-1)$ من خلال المقارنة نحصل على انه

* $f(x,y) = \frac{\cancel{2}x^2y}{\cancel{2}} + c(y)$ من المعادلة المعطاة   المدرس المساعد عمر ابراهيم الجبوري

* $\frac{\partial f}{\partial y} = x^2 + c'(y) = x^2 - 1 \Rightarrow c'(y) = -1$ by $\int$

$\therefore c'(y) = -y \Rightarrow f(x,y) = x^2y - y$

* $x^2y - y = C$ The general solution.

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② $\left(3x^2 + 2y + 2\cosh(2x+3y)\right)dx + \left(2x + 2y + 3\cosh(2x+3y)\right)dy = 0$

من خلال المقارنة مع الصيغة العامة نحصل على

$M(x,y) = 3x^2 + 2y + 2\cosh(2x+3y) \rightarrow \frac{\partial M}{\partial y} = 0 + 2 + 2\sinh(2x+3y)\cdot 3$

$N(x,y) = 2x + 2y + 3\cosh(2x+3y) \rightarrow \frac{\partial N}{\partial x} = 2 + 0 + 3\sinh(2x+3y)\cdot 2$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore$ Exact D.E.

$$* \quad \frac{\partial f}{\partial x} = 3x^2 + 2y + 2\cosh(2x+3y)$$

$$\frac{\partial f}{\partial y} = 2x + 2y + 3\cosh(2x+3y)$$

by $\int$ of $\frac{\partial f}{\partial x} \Rightarrow f(x,y) = \int 3x^2 + 2y + 2\cosh(2x+3y)\, dx$

$$\therefore f(x,y) = \frac{3x^3}{3} + 2xy + \sinh(2x+3y) + C(y)$$

بالمعادلة $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = 0 + 2x + 3\cosh(2x+3y) + C'(y) = 2x + 2y + 3\cosh(2x+3y)$$

$$\therefore C'(y) = 2y \Rightarrow C(y) = \int 2y\, dy \Rightarrow C(y) = y^2$$

$$\therefore f(x,y) = x^3 + 2xy + y^2 + \sinh(2x+3y)$$

$$\therefore x^3 + 2xy + y^2 + \sinh(2x+3y) = C \quad \text{general Soln.}$$

③ $(x + \sin y)\, dx + (x\cos y - 2y)\, dy = 0$.

Sol:

$$\left.\begin{aligned} M(x,y) &= x + \sin y \longrightarrow \frac{\partial M}{\partial y} = \cos y \\ N(x,y) &= x\cos y - 2y \longrightarrow \frac{\partial N}{\partial x} = \cos y - 0 \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact}$$

$$* \quad \frac{\partial f}{\partial x} = x + \sin y \quad \& \quad \frac{\partial f}{\partial y} = x\cos y - 2y$$

$$* \quad f(x,y) = \int x + \sin y\, dx \Rightarrow f(x,y) = \frac{1}{2}x^2 + x\sin y + C(y).$$

$$* \quad \frac{\partial f}{\partial y} = 0 + x\cos y + C'(y) = x\cos y - 2y \Rightarrow C'(y) = -2y$$

$C(y) = \int -2y\, dy \Rightarrow C(y) = -y^2$

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$\therefore f(x,y) = \frac{1}{2}x^2 + x\sin y - y^2$

$\therefore \frac{1}{2}x^2 + x\cdot\sin y - y^2 = C$ The general solution.

④ Non-Exact Differential Equation.

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ this eq. is non-exact.

- To convert it into exact eq, we must multiply it by integrating factor (I.F), Hence.

$M(x,y)\,dx + N(x,y)\,dy = 0. \quad * \mu(x,y).$

$\Rightarrow \mu M(x,y)\,dx + \mu N\,dy = 0.$, this eq is exact if.

$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$

- The integrating factor may be a function of x only, function of y only or function of both x and y

- The integrating factor can be found by :-

① If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x only,

$\mu(x,y)$ is a function of x only, $\mu = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$

② If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of y only. $\mu(x,y)$ is a function of y only, $\mu = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) dy}$

③ If $\mu(x,y)$ is a function of x & y, then partial diff. eqs. must be used.

Ex] Solve the differential eq. $(x+3y^2)dx+2xy\,dy=0$.

$M(x,y)=x+3y^2 \Rightarrow \frac{\partial M}{\partial y}=6y$

$N(x,y)=2xy \Rightarrow \frac{\partial N}{\partial x}=2y$

$\left.\right\} \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ Non-exact

check, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2xy}(6y-2y)=\frac{2}{x}$ (function of x only)

$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{x+3y^2}(6y-2y)$ حل لأنه دالة لـ $y.x$

$\therefore \mu = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiply the given eq. by the I.F $\mu = x^2$

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$\therefore x^2(x+3y^2)dx + x^2(2xy)dy = 0$.

$\therefore M(x,y)=x^3+3x^2y^2 \Rightarrow \frac{\partial M}{\partial y}=6x^2y$

$N(x,y)=2x^3y \Rightarrow \frac{\partial N}{\partial x}=6x^2y$

$\left.\right\} \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$\therefore$ Exact eqs.

$\therefore \frac{\partial f}{\partial x} = x^3 + 3x^2y^2 \quad \& \quad \frac{\partial f}{\partial y} = 2x^3y.$

$\therefore f(x,y) = \int x^3 + 3x^2y^2 \, dx \Rightarrow f(x,y) = \frac{x^4}{4} + x^3y^2 + C(y)$

$\therefore \frac{\partial f}{\partial y} = 0 + 2x^3y + C'(y) = 2x^3y \Rightarrow C'(y) = 0.$

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$C(y) = \int 0 \, dy \Rightarrow C(y) = C_1$

$\therefore f(x,y) = \frac{x^4}{4} + x^3y^2 + C_1 \Rightarrow \boxed{\frac{1}{4}x^4 + x^3y^2 = C}$

Ex} Solve $(y+2x)dx + x(y+x+1)\,dy = 0.$

Sol} $M(x,y) = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 1$

$N(x,y) = xy + x^2 + x \Rightarrow \frac{\partial N}{\partial x} = y + 2x + 1$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ — not exact

check $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1-(y+2x+1)}{xy+x^2+x}$ isn't function of x only

check $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1-(y+2x+1)}{y+2x} = \frac{\cancel{1} - y - 2x - \cancel{1}}{y+2x} = \frac{-(y+2x)}{(y+2x)} = -1$

$\therefore$ function of y only.

$\therefore \mu = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)} = e^{-\int -1\,dy} = e^{\int dy} = e^{y}$

$\therefore$ multiply the given eq. by I.F. (e^{y})

$\therefore e^{y}(y+2x)\,dx + xe^{y}(y+x+1)\,dy = 0.$

$\therefore M = ye^{y} + 2xe^{y} \qquad \therefore \frac{\partial M}{\partial y} = y \cdot e^{y} \cdot 1 + e^{y} \cdot 1 + 2xe^{y}(1)$

$N = xye^{y} + x^{2}e^{y} + xe^{y} \qquad \frac{\partial N}{\partial x} = ye^{y} + 2xe^{y} + e^{y}$

$M = \frac{\partial f}{\partial x} \Rightarrow f(x,y) = \int ye^{y} + 2xe^{y}\,dx.$

$f(x,y) = xye^{y} + x^{2}e^{y} + C(y).$ but $\frac{\partial f}{\partial y} = N$

$\therefore \frac{\partial f}{\partial y} = x(y * e^{y}(1) + e^{y} \times 1) + x^{2}e^{y}(1) + C'(y)$

$= \cancel{xye^{y}} + \cancel{xe^{y}} + \cancel{x^{2}e^{y}} + C'(y) = \cancel{xye^{y}} + \cancel{x^{2}e^{y}} + \cancel{xe^{y}}.$

$\therefore C'(y) = 0 \Rightarrow C(y) = \int 0\,dy \Rightarrow C(y) = C_1$

$\therefore f(x,y) = xye^{y} + x^{2}e^{y} + C_1$

$\therefore \boxed{xye^{y} + x^{2}e^{y} = C}$ G. S.

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(5) Linear Differential Equations.

The general form is $\frac{dy}{dx} + P(x)y = Q(x)$.

- To solve this type we must find the Integrating factor $V(x) = e^{\int p(x)dx}$

- The general Solution is $y = \frac{1}{V(x)} \cdot \int v(x) Q(x) dx$

Ex: Solve the following linear diff. eqs.

(1) $xy' - 3y = x^2$

Sol: Firstly we must write it in the general form

by $\div x \Rightarrow y' - \frac{3}{x} y = x$

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$\therefore P(x) = \frac{-3}{x}$, $Q(x) = x$.

$v(x) = e^{\int p(x) dx} = e^{\int \frac{-3}{x} dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3}$

$\therefore y = \frac{1}{v(x)} \int v(x) Q(x) dx \Rightarrow y = \frac{1}{1/x^3} \int \frac{1}{x^3} \cdot x \, dx$

$y = x^3 \int \frac{1}{x^2} dx = x^3 \cdot \left[\frac{-1}{x} + c\right]$ $\therefore \boxed{y = cx^3 - x^2}$

general solution

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② $\dfrac{dy}{dx} = x(2y+1)$

sol} $\dfrac{dy}{dx} = 2xy + x \Rightarrow \dfrac{dy}{dx} - 2xy = x$

L.D.E with $p(x) = -2x$ & $Q(x) = x$.

$v(x) = e^{\int p(x)dx} = e^{\int -2x\,dx} = e^{-x^2} = \dfrac{1}{e^{x^2}}$

تكامل دالة e مع توفر المشتقة

$y = \dfrac{1}{v(x)} \int p(x)\, Q(x)\, dx = \dfrac{1}{e^{x^2}} \int e^{-x^2} \cdot x\, dx$

$y = \dfrac{1}{e^{x^2}} \left(\dfrac{-1}{2} e^{-x^2} + c \right) \Rightarrow y = \dfrac{-1}{2} e^{-2x^2} + c e^{-x^2}$

③ $(1+x^2)\dfrac{dy}{dx} + xy + x^3 + x = 0$

devided by $(1+x^2) \Rightarrow \dfrac{dy}{dx} + \dfrac{x}{1+x^2} y = \dfrac{-x^3 - x}{1+x^2}$

$\dfrac{dy}{dx} + \dfrac{x}{1+x^2} y = \dfrac{-x\cancel{(1+x^2)}}{\cancel{(1+x^2)}} \Rightarrow p(x) = \dfrac{x}{1+x^2}, \; Q(x) = -x$

$v(x) = e^{\int p(x)dx} = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(1+x^2)} =$

$y = \dfrac{1}{v(x)} \int v(x)\, Q(x) = \dfrac{1}{\sqrt{1+x^2}} \int \sqrt{1+x^2} \cdot x\, dx$

$y = \left(\dfrac{-1}{\cancel{2}} (1+x^2)^{3/2} * \dfrac{\cancel{2}}{3} + c \right) \dfrac{1}{\sqrt{1+x^2}}$

توفير مشتقة داخل القوس

$$\therefore\ y = \frac{-1}{3}(1+x^2) + c\,(1+x^2)^{\frac{-1}{2}}$$

④ $(y - x + xy\cot x)\,dx + x\,dy = 0$

Sol} $\div\ x\,dx \Rightarrow \frac{dy}{dx} + \frac{y}{x} - \frac{\cancel{x}}{\cancel{x}} + \frac{\cancel{x}y}{\cancel{x}}\cot x = 0$

مشتقتها

$$\therefore\ \frac{dy}{dx} + \left(\frac{1 - x\cot x}{x}\right) y = 1 \qquad \left\{\ \int \cot x = \int \frac{\cos x}{\sin x}\,dx\right.$$

دالتها

$$\therefore\ P(x) = \frac{1 - x\cot x}{x} \qquad Q(x) = 1$$

$$v(x) = e^{\int P(x)\,dx} = e^{\int \frac{1 - x\cot x}{x}\,dx} = e^{\left[\int \frac{1}{x}dx - \int \cot x\,dx\right.}$$

$$= e^{\ln|x| - \ln|\sin x|} = e^{\ln\left|\frac{x}{\sin x}\right|} = \frac{x}{\sin x}$$

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$$\therefore\ y = \frac{1}{v(x)} \int P(x)\,Q(x)\,dx$$

$$y = \frac{1}{\frac{x}{\sin x}} \int \frac{x}{\sin x}(1)\,dx \longrightarrow \text{integration by parts.}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{x}{\sin x}\,dx \longrightarrow \text{let } u = x \quad du = dx$$

$$dv = \frac{1}{\sin x}\,dx \to v = \int \frac{1}{\sin x}\,dx = \int \csc x\,dx = \ln|\csc x - \cot x|$$

$$= x\ln|\csc x - \cot x| -$$

$$\int \ln|\csc x - \cot x|\,dx$$

$$y = \frac{\sin x}{x}\left[x \ln|\csc x - \cot x| - \int \ln|\csc x - \cot x|\, dx + c\right]$$

لايجاد هذا التكامل يتوجب استخدام integration by parts مرة أخرى

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⑤ Bernoulle's Equations.

لاحظوا الفرق بين هذا النوع والمعادلة الخطية هو y^n

The general form. $\frac{dy}{dx} + p(x)\cdot y = Q(x)\, y^n$

Hence if $n=0$ [equation is linear].

$n=1$ [equation is Separable].

$n \neq 0 \;\&\; n \neq 1$, the solution is done by

Changining variable from y to u. This gives a differential equation in $(x, \; u)$ that is linear D.E

- Deviding the above standard form by y^n.

$$\therefore \frac{1}{y^n} \cdot \frac{dy}{dx} + P(x)\, y^{1-n} = Q(x) \quad \text{———} \; *$$

let $u = y^{1-n} \Rightarrow \frac{du}{dx} = (1-n)\cdot y^{-n} \cdot \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{y^n}{1-n} * \frac{du}{dx}$ Sub into the eq $*$

$$\frac{1}{\cancel{y^n}} \frac{\cancel{y^n}}{1-n} * \frac{du}{dx} + P(x)\, u = Q(x) \quad \text{Linear D.E.}$$

Ex} Find the following differential equations.

① $\frac{dy}{dx}-\frac{1}{x}y=xy^2$

Sol} Divided by $y^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{x}y^{-1}=x$.

let $u=y^{-1} \Rightarrow \frac{du}{dx}=-1y^{-2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=-y^2\frac{du}{dx}$

$\therefore \frac{1}{\cancel{y^2}}*-\cancel{y^2}\frac{du}{dx}-\frac{1}{x}u=x\,\}*-1 \Rightarrow \frac{du}{dx}+\underbrace{\frac{1}{x}}_{P(x)}u=\underbrace{-x}_{Q(x)}$

$v(x)=e^{\int P(x)dx}=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

$u=\frac{1}{v(x)}\int v(x)*Q(x)\,dx=\frac{1}{x}\int x*-x\,dx=\frac{1}{x}\int -x^2\,dx$

$\therefore u=\frac{1}{x}\left[-\frac{x^3}{3}+c\right] \Rightarrow u=\frac{-1}{x}*\frac{x^3}{3}+\frac{1}{x}c$ but $u=\frac{1}{y}$

$\therefore \frac{1}{y}=\frac{-x^2}{3}-\frac{c}{x}$ General solution.

② $\frac{dy}{dx}+\frac{1}{3}y=e^xy^4$

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Sol} Divided eq. by $y^4 \Rightarrow \frac{1}{y^4}\frac{dy}{dx}+\frac{1}{3}y^{-3}=e^x$ ---*

let $u=y^{-3} \Rightarrow \frac{du}{dx}=-3y^{-4}\frac{dy}{dx}$

$\therefore \frac{dy}{dx}=\frac{-1}{3}y^4$ Sub into eq *

$$\frac{1}{\cancel{y^4}} \cdot \frac{-1}{3} \cancel{y^4} \frac{du}{dx} + \frac{1}{3} u = e^x \Big] * -3$$

$$\frac{du}{dx} - u = -3e^x$$ linear eq. $P(x) = -1,\ Q(x) = -3e^x$

$$V(x) = e^{\int P(x)dx} = e^{\int -1 dx} = e^{-x}$$

$$u = \frac{1}{V(x)} \int (x)\, Q(x)\, dx \Rightarrow u = \frac{1}{e^{-x}} \int e^{-x} \cdot -3e^{x} dx$$

$$u = e^x \int -3\, dx \Rightarrow u = e^x(-3x + c)$$ but $u = y^{-3} = \frac{1}{y^3}$

$$\therefore \frac{1}{y^3} = e^x(c - 3x)$$

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③ $$\frac{dy}{dx} = y\cot x + y^3 \csc x$$

$$\frac{dy}{dx} - y\cot x = y^3 \csc x \quad \div y^3$$

$$\frac{1}{y^3}\frac{dy}{dx} - (\cot x)\, y^{-2} = \csc x$$

let $u = y^{-2}$ $\quad \frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{2} y^3$

$$\therefore \frac{1}{\cancel{y^3}} \cdot \frac{-1}{2} \cdot \cancel{y^3} \frac{du}{dx} - (\cot x) \cdot u = \csc x \Big\} * -2$$

$$\frac{du}{dx} + 2\cot x * u = -2\csc x$$ linear D.E.

$P(x) = 2\cot x \quad , \quad Q(x) = -2\csc x.$

$$V(x) = e^{\int P(x)dx} = e^{\int 2\cot x\, dx} = e^{\int \frac{2\cos x}{\sin x} dx} = e^{2\ln \sin x}$$

$$= e^{\ln(\sin^2 x)} = \sin^2 x$$

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$$u = \frac{1}{V(x)} \int V(x)\, Q(x)\, dx = \frac{1}{\sin^2 x} \int \sin^2 x \cdot -2\csc x\, dx$$

$$= \frac{1}{\sin^2 x} \int (\sin^2 x)\left(\frac{-2}{\sin x}\right) dx = \frac{1}{\sin^2 x} \int -2 \sin x\, dx$$

$$u = \frac{1}{\sin^2 x}\left[-2(-\cos x) + C\right] \quad \text{but } u = \frac{1}{y^2}$$

$$\therefore \frac{1}{y^2} = \frac{1}{\sin^2 x}\left[2\cos x + C\right] \Rightarrow y^2 = \frac{\sin^2 x}{2\cos x + C}.$$

Second order differential equations.

2^{nd} order differential eqs is one in which the highest order differential coefficient is the second. The eq. may contain other terms involving first order differ. coefficients in addition to functions of the independent and dependent variables. 2^{nd} order differential can be written as

$$F(x, y, y', y'') = 0 \quad \text{or} \quad y'' = f(x, y, y')$$

Classification of 2^{nd} order D.E

A) Special second order D.E (Non-linear)

1- The dependent variable missing (y missing).

2- The independent variable missing (x missing).

3- The independent variable & the 1^{st} derivative missing (x, y')

B) Linear $\boxed{y'' + a_1(x)y' + a_0(x)y = Q(x)}$

1- The coefficients are constant. (a_1, a_0 are constant)

2- The coefficients are functions of the independent variable. (a_1 or a_0) is not constant.

3- Homogeneous if the source term ($Q(x) = 0$) otherwise it called nonhomogeneous.

* The non-linear 2^{nd} order D.E of type y missing or x missing are called "Reducible 2^{nd} order" because it can be solved by reducing it into first order differential eqs.

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* The first two classes of Linear D.E are called non-homogeneous.

Examples on 2^{nd} order D.E.

1- 2^{nd} order y missing , $y'' = -2x(y')^2$, $y'' + xy' = x$

2- 2^{nd} order x missing $y'' = 2yy'$, $y\,y'' + 3(y')^2 = 0$.

$$\frac{d^2y}{dx^2} + y = 0.$$

3- 2^{nd} order, Linear with constant coefficient

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \quad , \quad \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 2x^2 + x \quad , \quad y'' - 3y' + y = \cos(3t)$$

4- 2^{nd} order, Linear with Variable Coefficient

$$(x-1)\frac{d^2y}{dx^2} + \sin x\,\frac{dy}{dx} + 2y = \cos x \quad , \quad 4xy'' + 6y' + y = 0.$$

$$y'' + 2ty' - \ln(t)\,y = e^{3t}.$$

- Non linear D.E with dependent variable is missing (y missing).

- The general form. $f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0$.

- The solution is determined by assuming:-

$$\frac{dy}{dx} = P \Rightarrow \frac{d^2y}{dx^2} = \frac{dP}{dx}$$

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والغرض من هذه الفرضية هو تحويل ما معادلة تفاضلية من الدرجة الثانية إلى معادلة تفاضلية من الدرجة الأولى وحلها بالطرق التي تم دراستها مسبقاً.

Ex] Solve the following O.D.E $y'' + xy' = x$

Sol]
$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x$$

let $\frac{dy}{dx} = P \Rightarrow \frac{d^2y}{dx^2} = \frac{dP}{dx}$

$$\therefore \frac{dP}{dx} + xP = x \quad (\text{This eq. is } 1^{st} \text{ order D.E}).$$

$$\therefore \frac{dP}{dx} = x - xP = x(1-P) \Rightarrow \frac{dP}{1-P} = x\,dx \quad (\text{Separable})$$

(الإشارة السالبة من توفير مشتقة المقام)

$$\text{by } \int \Rightarrow -\ln(1-P) = \frac{x^2}{2} + C \Rightarrow \ln(1-P)^{-1} = \frac{x^2}{2} + C$$

$$\therefore \ln\frac{1}{1-P} = \frac{x^2}{2} + C \quad \text{take exp. for both sides.}$$

$$\therefore \frac{1}{1-P} = \exp\left(\frac{x^2}{2} + C\right) \Rightarrow 1-P = \frac{1}{e^{\left(\frac{x^2}{2}+C\right)}}$$

$\therefore p = 1 - \dfrac{1}{e^{\left(\frac{x^2}{2}+c\right)}}$ but $p = \dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = 1 - \dfrac{1}{e^{\left(\frac{x^2}{2}+c\right)}} \Rightarrow dy = \left(1 - \dfrac{1}{e^{\left(\frac{x^2}{2}+c\right)}}\right) dx$ (Separable)

by $\int \Rightarrow \int dy = \int 1\, dx - \int \dfrac{1}{e^{\left(\frac{x^2}{2}+c\right)}}$

$e^{\frac{x^2}{2}+c} = e^{\frac{x^2}{2}} \cdot e^{c} = e^{\frac{x^2}{2}} \cdot C_1$

$\therefore y = x - C_1 \int e^{-\frac{x^2}{2}} dx + C_2$

Ex} Solve the eq. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 6x$

Sol} let $p = \dfrac{dy}{dx} \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$.

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$\therefore x\dfrac{dp}{dx} + 2p = 6x \;\}\div x \Rightarrow \dfrac{dp}{dx} + \dfrac{2}{x}p = 6$ [1st order linear]

$v(x) = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$

$\therefore p = \dfrac{1}{v(x)} \int v(x)\,Q(x)\,dx = \dfrac{1}{x^2}\int x^2 \cdot (6)\,dx = \dfrac{1}{x^2}\left[\dfrac{6x^3}{3} + c\right]$

$\therefore p = 2x + \dfrac{c}{x^2}$ but $p = \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = 2x + \dfrac{c}{x^2}$

$\therefore dy = \left(2x + \dfrac{c}{x^2}\right)dx$ by $\int \Rightarrow \boxed{y = \dfrac{2x^2}{2} - \dfrac{c}{x} + C_2}$ G.S.

- Non linear D.E. with independent variable is missing (x, missing)

- The general form $f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0.$

∴ The solution is determined by assuming:-

$\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = \frac{dp}{dy}\, p$ because.

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p \quad \text{[chain Rule].}$$

Ex} Solve the eq. $\frac{d^2y}{dx^2} + y = 0.$

Sol} let $\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot p$

∴ $p\frac{dp}{dy} + y = 0$ (1st order, separable).

$$p\frac{dp}{dy} = -y \Rightarrow p\,dp = -y\,dy \quad \text{by } \int \Rightarrow \frac{p^2}{2} = \frac{-y^2}{2} + c \Big\} * 2$$

∴ $p^2 = -y^2 + C_1$ but $p = \frac{dy}{dx} \Rightarrow \left(\frac{dy}{dx}\right)^2 = C_1 - y^2.$

∴ $\frac{dy}{dx} = \sqrt{C_1 - y^2}$ (1st order, separable) by $\int$.

$$\int \frac{dy}{\sqrt{C_1 - y^2}} = \int dx \Rightarrow x = \int \frac{dy}{\sqrt{C_1 - y^2}} \Rightarrow x = \sin^{-1}\left(\frac{y}{c}\right) + C_2$$

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Ex} Solve the eq. $y\,y'' = (y')^2$.

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S-1} let $p = \frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = p\frac{dp}{dy}$

$\therefore y \cdot p \cdot \frac{dp}{dy} = (p)^2 \quad \div p \Rightarrow y\frac{dp}{dy} = p$ (1^{st} order, Separable)

$\therefore \frac{dp}{p} = \frac{dy}{y}$ by $\int \Rightarrow \int\frac{dp}{p} = \int\frac{dy}{y} \Rightarrow \ln p = \ln y + \ln c$

by $\ln p = \ln(y \cdot c)$ by exp $\Rightarrow p = c\,y$

but $p = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = c \cdot y \Rightarrow \frac{dy}{y} = c\,dx \Rightarrow$ by $\int$.

$\ln y = c_1 x + c \Rightarrow y = \exp(c_1 x + c)$

- Homogeneous Linear eq. with Constant coefficients.

- The general form $a_2 y'' + a_1 y' + a_0 y = 0$

- The solution is done by using Differential operator (D-operator) where $\frac{dy}{dx} = Dy$, then using the Characteristic Equation.

$a_2 m^2 + a_1 m + a_0 = 0.$

معادلة من الدرجة الثانية تحل بالدستور او بالتجربة

حيث أن للمعادلة جذران هما m_1, m_2

- There will be three forms of the general solution according to the type of the root m_1, m_2.

① m_1 & m_2 are unequal real roots, the general solution is:-

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

② m_1 & m_2 are equal real roots $m_1 = m_2$, the general solution is:-

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

③ m_1 & m_2 are Complex or $m_1 = \alpha + i\beta$, $m_1 = \alpha - i\beta$, the general solution is:-

$$y = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$$

$$= e^{\alpha x} (c_1 \cos\beta x + c_2 \sin\beta x)$$

Ex} Solve the following diff. eqs.

① $2y'' - 5y' - 3y = 0$.

S-1} $2m^2 - 5m - 3 = 0$

$(2m+1)(m-3) = 0$

either $2m+1 = 0 \Rightarrow m_1 = \frac{-1}{2}$

or $m-3 = 0 \Rightarrow m_2 = 3$

} unequal real roots.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\therefore y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x} \quad G.S$$

(2) $y'' - 10y' + 25y = 0$

Sol: $m^2 - 10m + 25 = 0 \Rightarrow (m-5)(m-5) = 0.$

either $m-5=0 \quad m_1 = 5$
or $m-5=0 \quad m_2=5$ } equal real roots.

$\therefore y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

$\therefore y = c_1 e^{5x} + c_2 x e^{5x}$ G.S.

(3) $y'' + 4y' + 7y = 0.$

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Sol: $m^2 + 4m + 7 = 0$, $a=1, b=4 \;\&\; c=7.$ تحل بالدستور

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \mp \sqrt{4^2 - 4(1)(7)}}{2(1)} = \frac{-4 \mp \sqrt{-12}}{2}$$

$$m = \frac{-4 \mp 2\sqrt{3}\,i}{2} \Rightarrow m = -2 \mp \sqrt{3}\,i \Rightarrow m_1 = -2 + \sqrt{3}\,i,\; m_2 = -2 - \sqrt{3}\,i$$

$\therefore \alpha = -2, \; \beta = \sqrt{3}$

$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta(x))$

$y = e^{-2x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$ G.S.