The Chain Rule

The chain rule for functions of a single variable says that when $w = f(x)$ and $x = g(t)$, then $\frac{dw}{dt}$ can be calculated by the Formula $\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$

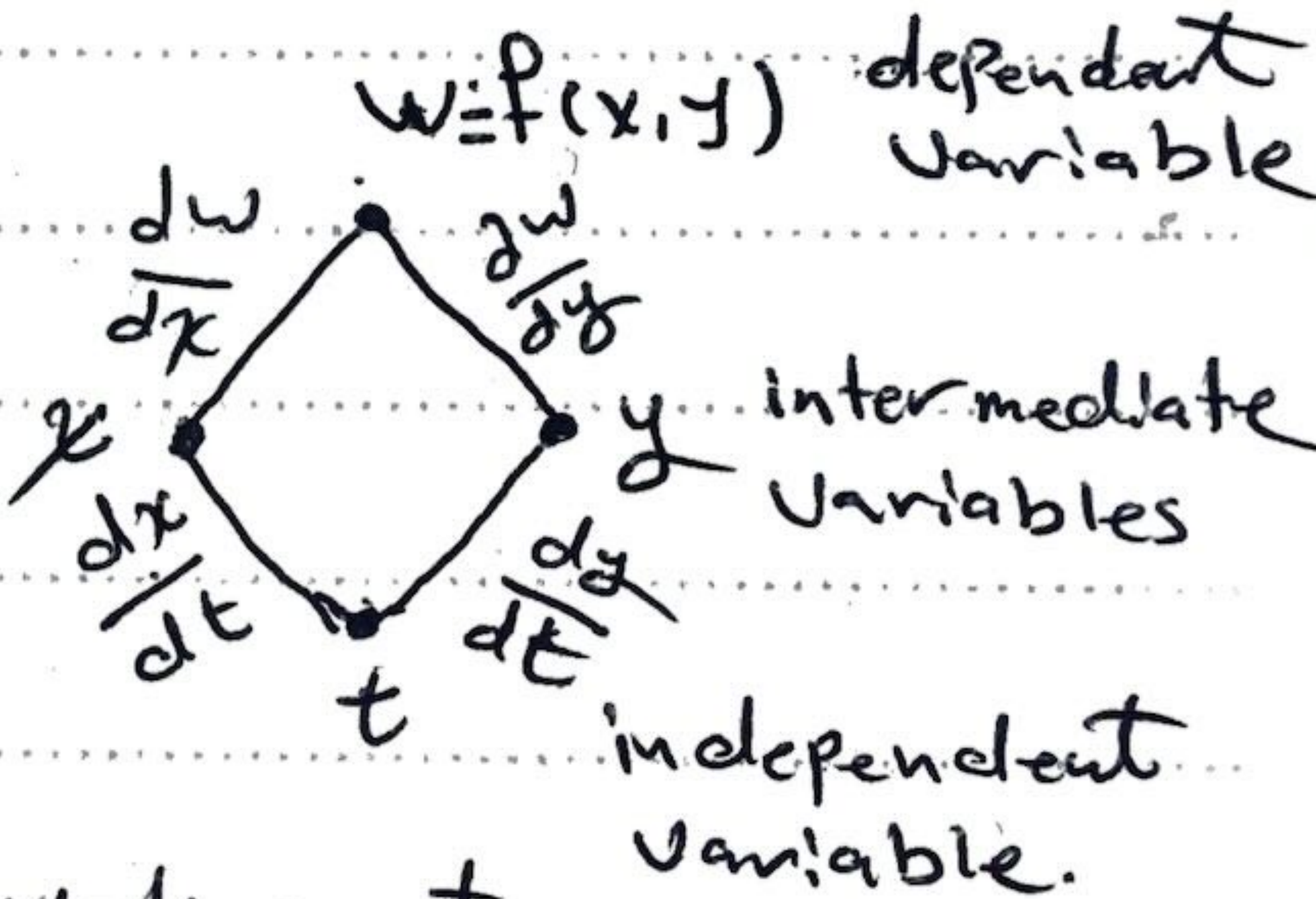
For functions of two variables, the chain Rule for function $w = f(x, y)$ when $x = x(t)$, $y = y(t)$, then

القانون الأول

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}}$$

حفظ

The branch diagram provides a convenient way to remember the Chain Rule.



Ex] Use the chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. what is the derivative's value at $t = \frac{\pi}{2}$?

Sol]

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial (xy)}{\partial x} \cdot \frac{d}{dt}(\cos t) + \frac{\partial (xy)}{\partial y} \cdot \frac{d}{dt}(\sin t)$$

$$= y \cdot (-\sin t) + x \cdot (\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

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$$\therefore \frac{dw}{dt} = -\sin^2 t + \cos^2 t$$
$$= \cos 2t$$

إيضاً نستطيع حل المسألة من خلال تعويض الدالتين x, y في الدالة w حيث ينتج لدينا w كدالة لـ t ونشتقها مباشرة

$$w = xy = \cos t \cdot \sin t = \frac{1}{2}\sin 2t$$

$$\therefore \frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{\cancel{2}}(\cos 2t)\cdot \cancel{2} = \cos 2t$$

at the given value of $t = \frac{\pi}{2}$

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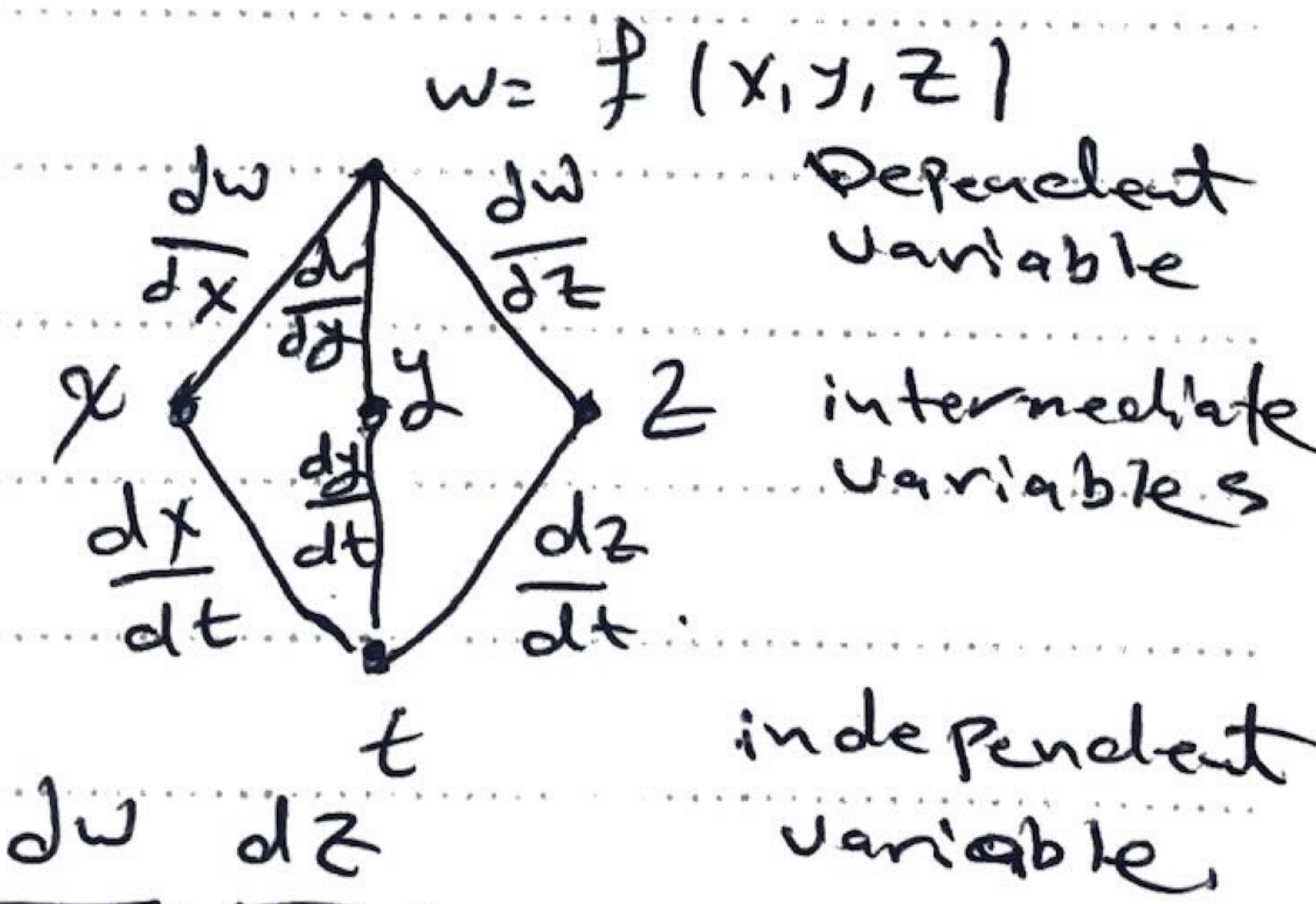
$$\left(\frac{dw}{dt}\right)_{t=\frac{\pi}{2}} = \cos\left(2\cdot\frac{\pi}{2}\right) = \cos\pi = -1$$

- Chain Rule can be applied to functions of three variables, if $w = f(x, y, z)$

$$\boxed{\frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt} + \frac{dw}{dy}\cdot\frac{dy}{dt} + \frac{dw}{dz}\cdot\frac{dz}{dt}}$$

حفظ
الحالة الثانية

Ex) Find $\frac{dw}{dt}$ if $w = xy + z$.
$x = \cos t$, $y = \sin t$, $z = t$
what is the derivative's values at $t = 0$?



Sol) $\frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt} + \frac{dw}{dy}\cdot\frac{dy}{dt} + \frac{dw}{dz}\cdot\frac{dz}{dt}$.

$$= \frac{d}{dx}(xy+z)\cdot\frac{d}{dt}(\cos t) + \frac{d}{dy}(xy+z)\cdot\frac{d}{dt}(\sin t) + \frac{d}{dz}(xy+z)\cdot\frac{d}{dt}(t)$$

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$$\frac{dw}{dt} = y(-\sin t) + x\cdot(\cos t) + 1\,(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= 1 + \cos 2t$$

at $t=0$ $\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos 2(0) = 1 + 1 = 2.$

ايضاً نستطيع ايجاد $\frac{dw}{dt}$ بتعويض دوال $x, y \,\&\, z$ في دالة w واشتقاقها مباشرة

$$w = xy + z = (\cos t)(\sin t) + t.$$

$$\boxed{\cos^2 t - \sin^2 t = \cos 2t}$$

$$\frac{dw}{dt} = (\cos t)(\cos t) + (\sin t)(-\sin t) + 1$$

$$= \cos^2 t - \sin^2 t + 1 = 1 + \cos 2t$$

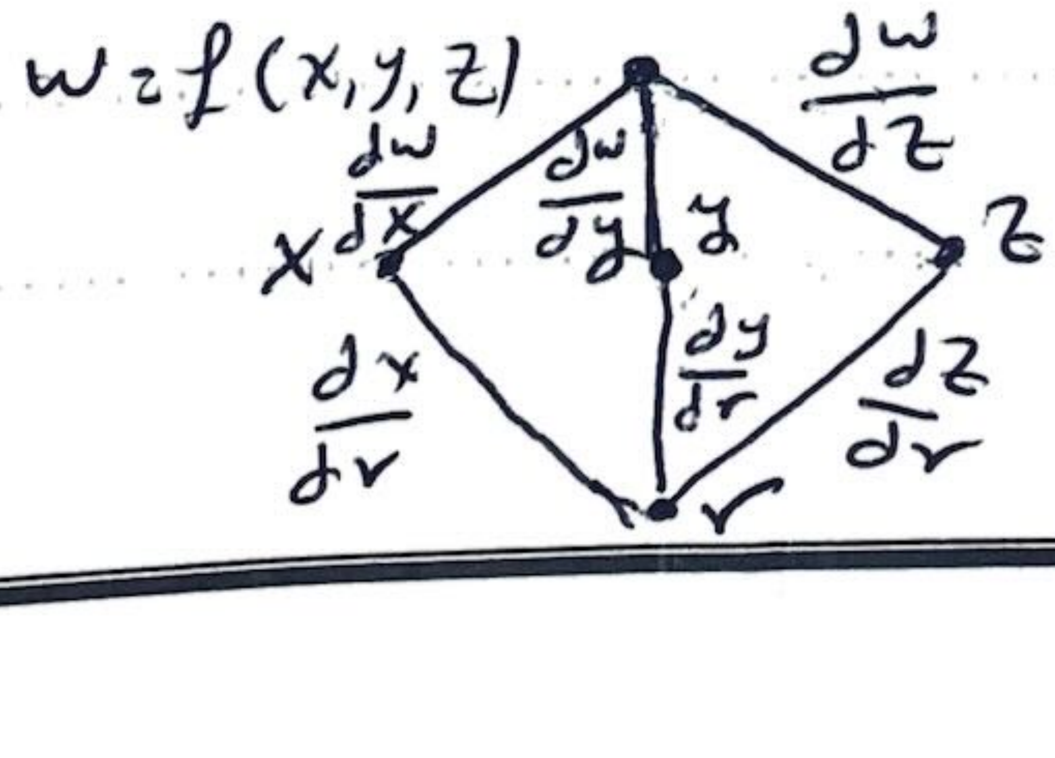
Chain Rule for two Independent variables and three Intermediate

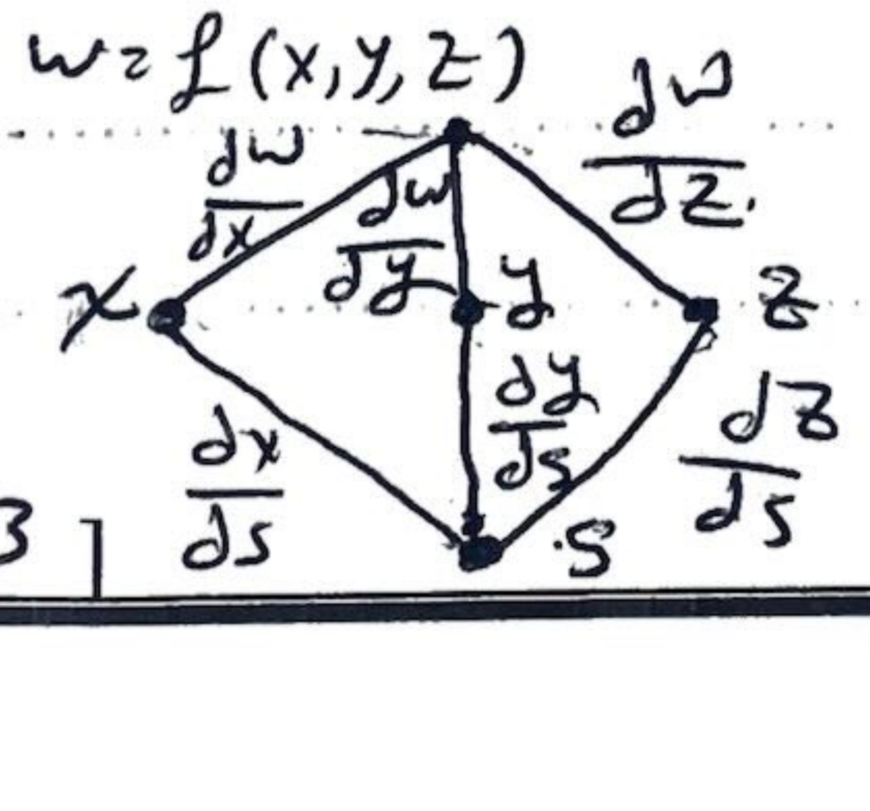
If $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$ & $z = K(r, s)$

then w has partial derivatives with respect to r & s

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$ الحالة الثالثة

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$





Ex} Express $\frac{dw}{dr}$ and $\frac{dw}{ds}$ in terms of r & s if

$w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$ & $z = 2r$

Sol}

$$\frac{dw}{dr} = \frac{dw}{dx}\cdot\frac{dx}{dr} + \frac{dw}{dy}\cdot\frac{dy}{dr} + \frac{dw}{dz}\cdot\frac{dz}{dr}$$

$$\frac{dw}{dr} = \frac{d}{dx}(x+2y+z^2)\cdot\frac{d}{dr}\left(\frac{r}{s}\right) + \frac{d}{dy}(x+2y+z^2)\cdot\frac{d}{dr}(r^2+\ln s) + \frac{d}{dz}(x+2y+z^2)\cdot\frac{d}{dr}(2r)$$

$$\therefore \frac{dw}{dr} = (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) = \frac{1}{s} + 4r + 4z$$

نعوض دالة Z لان في السؤال طلب المشتقة كدالة لـ r, s فقط

$$\therefore \frac{dw}{dr} = \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$$

$$\frac{dw}{ds} = \frac{dw}{dx}\cdot\frac{dx}{ds} + \frac{dw}{dy}\cdot\frac{dy}{ds} + \frac{dw}{dz}\cdot\frac{dz}{ds}$$

$$= \frac{d}{dx}(x+2y+z^2)\cdot\frac{d}{ds}\left(\frac{r}{s}\right) + \frac{d}{dy}(x+2y+z^2)\cdot\frac{d}{ds}(r^2+\ln s) + \frac{d}{dz}(x+2y+z^2)\cdot\frac{d}{ds}(2r)$$

$$\frac{dw}{ds} = (1)\cdot\left(-\frac{r}{s^2}\right) + (2)\cdot\left(\frac{1}{s}\right) + (2z)(0)$$

$$= -\frac{r}{s^2} + \frac{2}{s}$$

Ex} Express $\frac{dw}{dr}$ and $\frac{dw}{ds}$ in terms of r & s if

$w = x^2 + y^2$, $x = r - s$ & $y = r + s$

Sol} $\frac{dw}{dr} = \frac{dw}{dx}\cdot\frac{dx}{dr} + \frac{dw}{dy}\cdot\frac{dy}{dr}$

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$$\frac{dw}{dr}=\frac{d}{dx}(x^2+y^2)\cdot\frac{d}{dr}(r-s)+\frac{d}{dy}\cdot(x^2+y^2)\cdot\frac{d}{dr}\cdot(r+s)$$

$$=(2x)\cdot(1)+(2y)(1)=2x+2y=2(r-s)+2(r+s)$$

$$\frac{dw}{dr}=2r-2s+2r+2s=\boxed{4r}$$

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$$\frac{dw}{ds}=\frac{dw}{dx}\cdot\frac{dx}{ds}+\frac{dw}{dy}\cdot\frac{dy}{ds}$$

$$=\frac{d}{dx}(x^2+y^2)\cdot\frac{d}{ds}(r-s)+\frac{d}{dy}(x^2+y^2)\cdot\frac{d}{ds}(r+s).$$

$$=(2x)\cdot(-1)+(2y)(1)=-2x+2y=-2(r-s)+2(r+s)$$

$$=-\cancel{2r}+2s+\cancel{2r}+2s=\boxed{4s}$$

والآن تستطيع حل المسألة بطريقة أخرى وهي تعويض دالتي x,y في دالة w

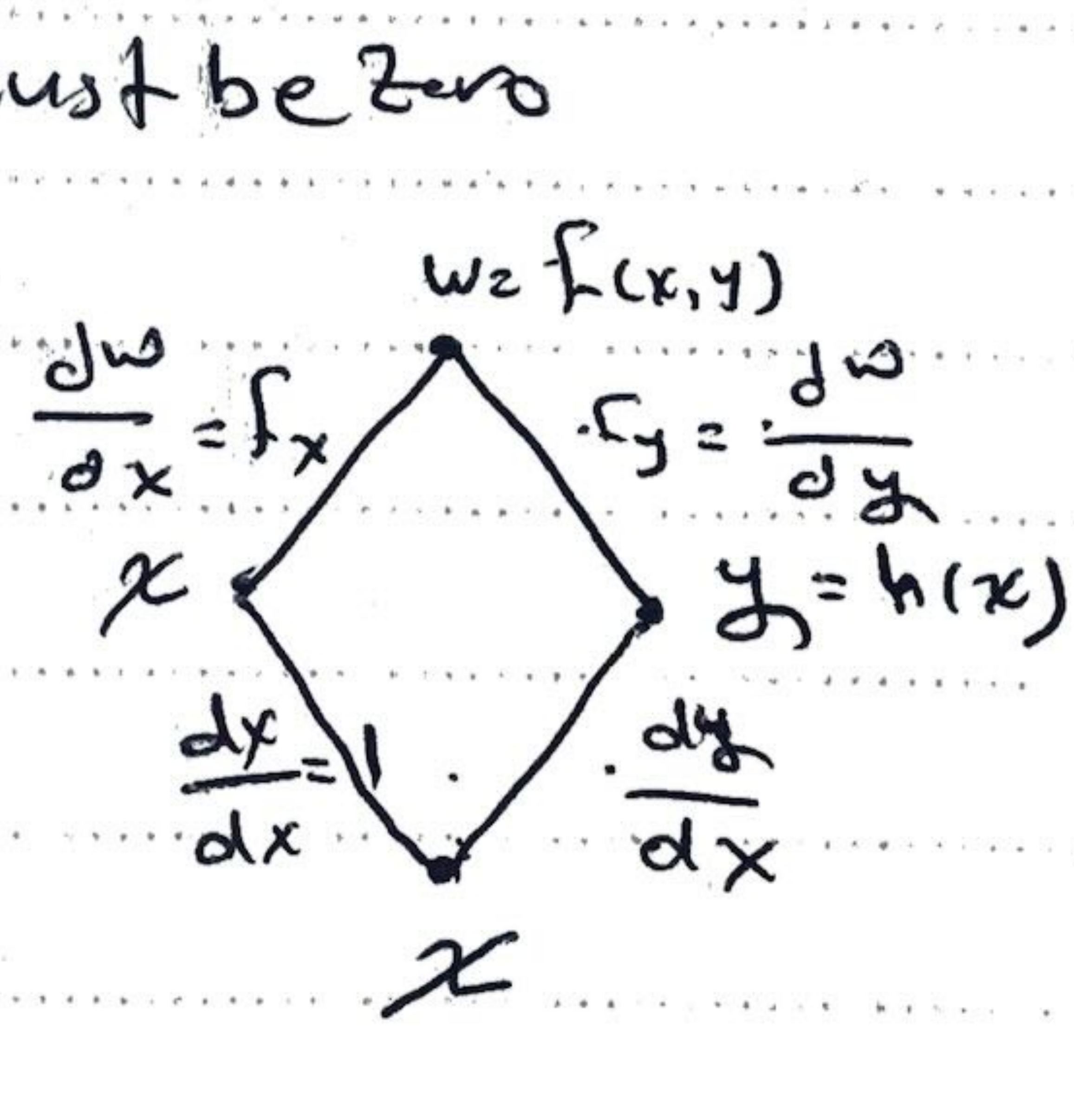
$$w=x^2+y^2=(r-s)^2+(r+s)^2=r^2-\cancel{2rs}+s^2+r^2+\cancel{2rs}+s^2$$

$$\therefore w=2s^2+2r^2 \Rightarrow \frac{dw}{dr}=4r \;\&\; \frac{dw}{ds}=4s$$

Implicit Differentiation with Chain Rule. الحالة الرابعة

If $w=f(x,y)=0$, the derivative $\frac{dw}{dx}$ must be zero

$$0=\frac{dw}{dx}=f_x\frac{dx}{dx}+f_y\frac{dy}{dx}$$



$$\therefore 0=f_x+f_y\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=-\frac{f_x}{f_y}$$

Ex] Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$

Sol] Take $F(x,y) = y^2 - x^2 - \sin xy$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}, \quad F_x = -2x - (\cos xy)\,y, \quad F_y = 2y - (\cos xy)\,x.$$

$$\therefore \frac{dy}{dx} = -\frac{-2x - y\cos xy}{2y - x\cdot\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

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Ex] Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at $(0,0,0)$ if $x^3 + z^2 + y e^{xz} + z\cos y = 0$

Sol] let $F(x,y,z) = x^3 + z^2 + y e^{xz} + z\cos y$

$$F_x = 3x^2 + zye^{xz}; \quad F_y = e^{xz} - z\sin y \quad \& \quad F_z = 2z + xye^{xz} + \cos y$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{3x^2 + zy\cdot e^{xz}}{2z + xye^{xz} + \cos y}, \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}$$

at point $(0,0,0) \longrightarrow \frac{\partial z}{\partial x} = -\frac{3(0)^2 + (0)(0)e^{0}}{0 + 0e^{0} + \cos 0} = \frac{-0}{1} = 0$

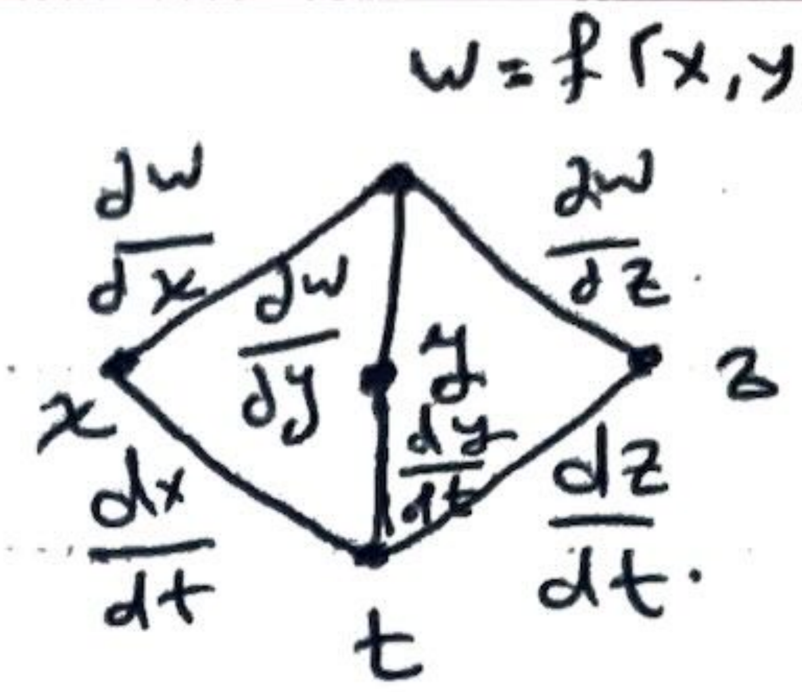
$$\frac{\partial z}{\partial y} = \frac{-(e^{0} - 0\sin 0)}{2(0) + 0e^{0} + \cos 0} = \frac{-1}{1} = -1$$

Ex] evaluate $\frac{dw}{dt}$ at the given value of t for أمثلة متنوعة

① $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$ and $t = 3$.

Sol] واضح من المعطيات أن هذه المسألة فيها متغير معتمد واحد وهو w
وثلاثة متغيرات وسطية x, y, z ومتغير مستقل واحد وهو t كما موضح بالمخطط

$$\therefore \frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt} + \frac{dw}{dy}\cdot\frac{dy}{dt} + \frac{dw}{dz}\cdot\frac{dz}{dt}.$$



الآن نستطيع إيجاد هذه الحدود متفرقة أو في سياق واحد

$$\frac{dw}{dt} = \frac{d}{dx}\left(\frac{x}{z}+\frac{y}{z}\right)\cdot\frac{d}{dt}(\cos^2 t) + \frac{d}{dy}\left(\frac{x}{z}+\frac{y}{z}\right)\cdot\frac{d}{dt}(\sin^2 t) + \frac{d}{dz}\left(\frac{x}{z}+\frac{y}{z}\right)\cdot\frac{d}{dt}\left(\frac{1}{t}\right)$$

$$= \frac{1}{z} * 2(\cos t)(-\sin t) + \frac{1}{z}\cdot 2(\sin t)\cos t + \frac{-(x+y)}{z^2}\cdot\frac{-1}{t^2}$$

$$= -\cancel{\frac{2}{z}\cos t\cdot\sin t} + \cancel{\frac{2}{z}\cos t\cdot\sin t} + \frac{x+y}{t^2\cdot z^2} = \frac{x+y}{t^2\cdot z^2} = \frac{\cos^2 t+\sin^2 t}{t^2\cdot\left(\frac{1}{t}\right)^2}$$

$$\frac{dw}{dt} = \cos^2 t + \sin^2 t = 1 \Rightarrow \boxed{\left.\frac{dw}{dt}\right)_{t=3} = 1}$$

أيضاً يمكن إيجاد المشتقة من خلال تعويض دوال x,y,z في دالة w ومن ثم اشتقاقها

$$w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\frac{1}{t}} + \frac{\sin^2 t}{\frac{1}{t}} = t(\cos^2 t + \sin^2 t) = t$$

$$\therefore w = t \quad \boxed{\frac{dw}{dt} = 1}$$

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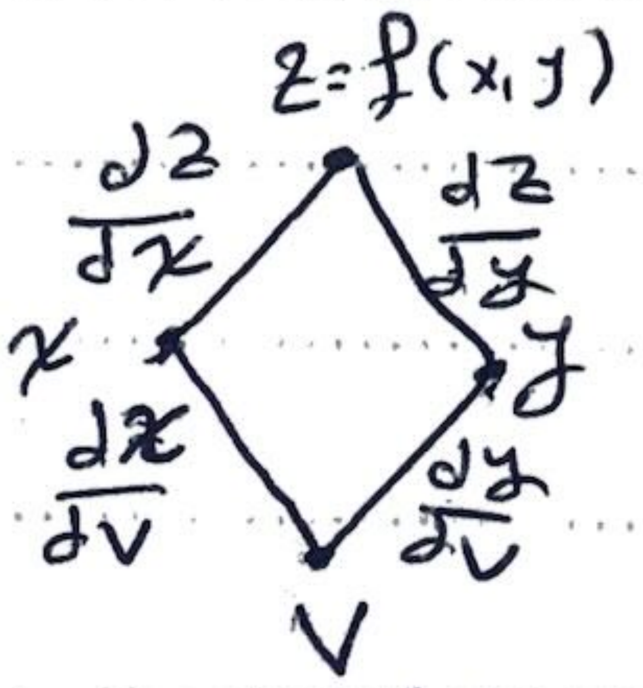
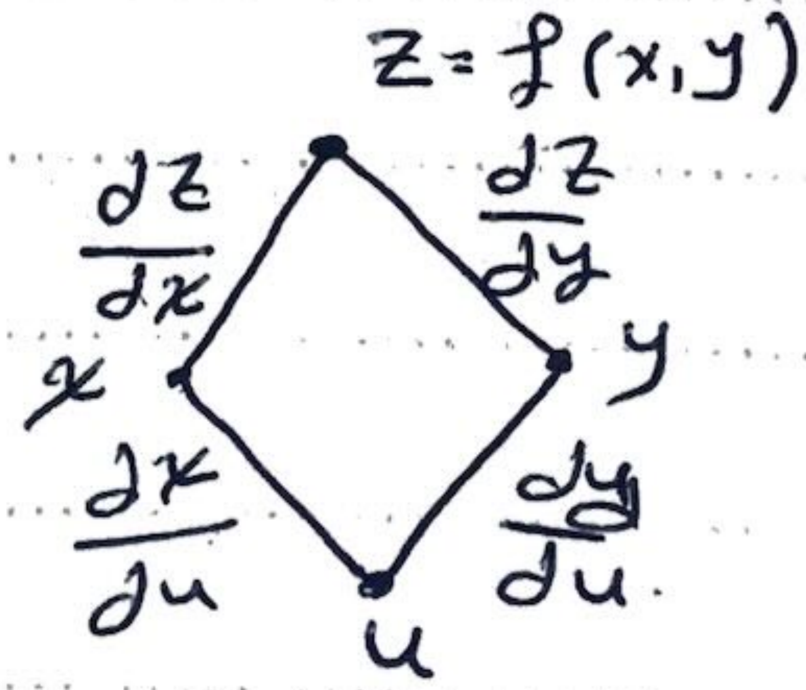
Ex] Evaluate $\frac{dz}{du}$ & $\frac{dz}{dv}$ at the given point

① $z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = u\cos v$, $y = u\sin v$ & $(u,v) = \left(1.3, \frac{\pi}{6}\right)$

$$\frac{dz}{du} = \frac{dz}{dx}\cdot\frac{dx}{du} + \frac{dz}{dy}\cdot\frac{dy}{du}$$





$$\frac{dz}{dx} = \frac{1}{\left(\frac{x}{y}\right)^2+1} * \frac{1}{y}$$

$$\frac{dx}{du} = \cos v, \quad \frac{dz}{dy} = \frac{1}{\left(\frac{x}{y}\right)^2+1} * \frac{-x}{y^2}, \quad \frac{dy}{du} = \sin v.$$

$$\therefore \frac{dz}{du} = \frac{\frac{1}{y}}{\left(\frac{x}{y}\right)^2+1} * \cos v + \frac{-\frac{x}{y^2}}{\left(\frac{x}{y}\right)^2+1} * \sin v.$$

$$= \frac{\frac{1}{y}}{\frac{x^2+y^2}{y^2}} \cdot \cos v + \frac{-\frac{x}{y^2}}{\frac{x^2+y^2}{y^2}} \cdot \sin v = \frac{y\cos v}{x^2+y^2} - \frac{x\sin v}{x^2+y^2}$$

$$= \frac{u\sin v \cdot \cos v}{u^2\sin^2 v + u^2\cos^2 v} - \frac{u\cos v \cdot \sin v}{u^2\sin^2 v + u^2\cos^2 v} =$$

$$= \frac{u\sin v \cdot \cos v - u\sin v\cos v}{u^2(\sin^2 v + \cos^2 v)} = \frac{0}{u^2} = 0$$

$$\frac{dz}{dv} = \frac{dz}{dx}\cdot\frac{dx}{dv} + \frac{dz}{dy}\cdot\frac{dy}{dv}$$

$$\frac{dx}{dv} = -u\sin v \quad , \frac{dy}{dv} = u\cos v$$

$$\frac{dz}{dv} = \frac{\frac{1}{y}}{\left(\frac{x}{y}\right)^2+1}(-u\sin v) + \frac{-\frac{x}{y^2}}{\left(\frac{x}{y}\right)^2+1}(u\cos v)$$

$$= \frac{\frac{1}{y}}{\frac{x^2+y^2}{y^2}} \cdot (-u\sin v) + \frac{-\frac{x}{y^2}}{\frac{x^2+y^2}{y^2}}(u\cos v)$$

$$= \frac{-yu\sin v}{x^2+y^2} - \frac{xu\cos v}{x^2+y^2} = \frac{-(u\sin v)u\sin v - (u\cos v)\cdot u\cos v}{u^2\cos^2 v + u^2\sin^2 v}$$

$$= \frac{-u^2(\sin^2 v + \cos^2 v)}{u^2(\sin^2 v + \cos^2 v)} = -1 \quad , \frac{dz}{dv} = -1 \quad at : (1, 3, \tfrac{\pi}{6}).$$

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Ex} Assuming that the equation below define y as differentiable function of x, find the value of $\frac{dy}{dx}$ at the given point

① $x^2+xy+y^2-7=0$ at $(1,2)$

Sol} let $F(x,y)=x^2+xy+y^2-7=0$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}, \quad F_x=2x+y, \quad F_y=x+2y$$

$$\frac{dy}{dx}=-\frac{2x+y}{x+2y} \text{ at } (1,2) \quad \left.\frac{dy}{dx}\right|_{(1,2)}=-\frac{2(1)+2}{1+2(2)}=\frac{-4}{5}$$

② $x\,e^{y}+\sin xy+y-\ln 2=0$ at $(0,\ln 2)$

Sol} let $F(x,y)=xe^{y}+\sin xy+y-\ln 2=0$

$$F_x=e^{y}+(\cos xy)y, \quad F_y=xe^{y}+(\cos xy)x+1$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{e^{y}+y\cos xy}{xe^{y}+x\cos xy+1}=-\frac{e^{\ln 2}+\ln 2\cos(0\times\ln 2)}{0\times e^{\ln 2}+(0)(\cos 0\times\ln 2)+1}$$

$$=\frac{-(2+\ln 2)}{0+0+1}=-(2+\ln 2)$$

Ex} Find the values of $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at the given points

① $z^3-xy+yz+y^3-2=0$, $(1,1,1)$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y}=\frac{-F_y}{F_z}, \quad \text{let } f(x,y,z)=z^3+xy+yz+y^3-2=0.$$

$$F_x=0+y=y, \quad F_y=-x+z+3y^2, \quad F_z=3z^2+y$$

$$\frac{\partial z}{\partial x}=-\frac{y}{3z^2+y}=\frac{1}{3(1)^2+1}=\frac{1}{4}$$

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$$\frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} = -\frac{-1+1+3(1)^2}{3(1)^2+1} = -\frac{3}{4}$$

② $\sin(x+y)+\sin(y+z)+\sin(x+z)=0 \quad (\pi,\pi,\pi)$

let $f(x,y,z) = \sin(x+y)+\sin(y+z)+\sin(x+z)=0.$

$$\frac{dz}{dx} = -\frac{F_x}{F_z}, \quad \frac{dz}{dy} = -\frac{F_y}{F_z}.$$

$F_x = \cos(x+y)+\cos(x+z)$, $F_y = \cos(x+y)+\cos(y+z)$

$F_z = \cos(y+z)+\cos(x+z)$

$$\frac{dz}{dx} = -\frac{\cos(x+y)+\cos(x+z)}{\cos(y+z)+\cos(x+z)} = -\frac{\cos 2\pi+\cos 2\pi}{\cos 2\pi+\cos 2\pi} = -1$$

$$\frac{dz}{dy} = -\frac{\cos(x+y)+\cos(y+z)}{\cos(y+z)+\cos(x+z)} = -\frac{\cos 2\pi+\cos 2\pi}{\cos 2\pi+\cos 2\pi} = -1$$

③ $xe^y+ye^z+2\ln x = 2+3\ln 2 \quad (1, \ln 2, \ln 3)$

let $f(x,y,z) = xe^y+ye^z+2\ln x-2+3\ln 2=0$

$$\frac{dz}{dx} = -\frac{F_x}{F_z}, \quad \frac{dz}{dy} = -\frac{F_y}{F_z}$$

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$F_x = e^y+2\cdot\frac{1}{x}$, $F_y = xe^y+e^z$, $F_z = ye^z$

$$\frac{dz}{dx} = -\frac{e^y+\frac{2}{x}}{ye^z} = -\frac{e^{\ln 2}+\frac{2}{1}}{(\ln 2)e^{\ln 3}} = -\frac{2+2}{3\ln 2} = \frac{-4}{3\ln 2}$$

$$\frac{dz}{dy} = -\frac{\cancel{x}e^y+e^z}{ye^z} = -\frac{1e^{\ln 2}+e^{\ln 3}}{(\ln 2)\cdot e^{\ln 3}} = -\frac{2+3}{3\ln 2} = -\frac{5}{3\ln 2}$$